REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	29 Jun 98	

4. TITLE AND SUBTITLE
OPTIMAL TWO STAGE PROCEDURES FOR ESTIMATING FUNCTIONS OF PARAMETERS IN RELIABILITY AND QUEUENING MODELS

5. FUNDING NUMBERS

6. AUTHOR(S)
Kevin Edward Burns

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of North Carolina

8. PERFORMING ORGANIZATION REPORT NUMBER
98-011D

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
THE DEPARTMENT OF THE AIR FORCE
AFIT/CIA, BLDG 125
2950 P STREET
WPAFB OH 45433

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION AVAILABILITY STATEMENT
Unlimited distribution
In Accordance With AFI 35-205/AFIT Sup 1

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

19980708 012

14. SUBJECT TERMS	15. NUMBER OF PAGES
	68
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT

OPTIMAL TWO STAGE PROCEDURES FOR ESTIMATING FUNCTIONS OF PARAMETERS IN RELIABILITY AND QUEUEING MODELS

by

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B.S., The United States Air Force Academy, 1988

M.S., The University of North Carolina, 1993

A Dissertation Submitted to the Graduate Faculty

of The University of Georgia in Partial Fulfillment

of the

Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

1998

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ACKNOWLEDGMENTS

I would like to thank some of the many people who helped make this dissertation possible. I couldn't possibly list all of the people who have helped and inspired me during my academic career, but I would like to mention those that had a direct impact on this dissertation.

First, I would like to thank my major professor, Dr. Andrew F. Seila. It is difficult to express and impossible to overstate how much of an impact he had on this project. He suggested the topic and was very instrumental in helping me form it into a workable problem. In addition, he diligently read and reread draft after draft and chapter after chapter always providing insight, thoughtful analysis, and suggestions for improvement. He never seemed to tire of patiently teaching me the art of research. Dr. Seila was a tremendous help to me in other areas than just research. Because I was working under a tight time constraint of three years, I could not afford many large bumps in the road. Dr. Seila smoothed over many bumps for me. Before I even arrived, he took the time to map out a plan of study that would fit my schedule. He went out of his way to teach me several summer courses on his own time so that I could finish on time. I will always be grateful for his assistance.

I would also like to thank the other members of my committee. Dr. T N Sriram spent many hours helping me work out technical details in the proofs of the theorems. He also helped me see the problem from a broader perspective. Dr. Robert Lund was of great assistance in helping me learn about estimating parameters from queueing systems. Dr. Ralph Steuer and Dr. Betty Whitten were great readers.

I want to thank Dr. Ted Shifrin who was a great mathematical consultant on Taylor Series expansions, and improper integrals.

Completing or even starting this program would not have been possible without the United States Air Force and Colonel Daniel Litwhiler, department head of Mathematical Sciences at the United States Air Force Academy. Colonel Litwhiler has had a tremendous impact on my Air Force and academic careers. From providing leadership when I was a mathematics major at the Air Force Academy to sponsoring my advanced degree in mathematics at UNC and selection as an instructor in his department to finally sponsoring this program, I will always be grateful to Colonel Litwhiler.

There are countless people who indirectly impacted my success in this program. It is hard to measure how much it helps when people believe in you. My parents Mr. Herbert Burns and Mrs. Rose Burns have believed in me all my life. They never doubted once whether I could accomplish this goal. Thanks Mom and Dad!

Last, but by no means least, I would like to thank my lovely wife, Christina. No one could have been more patient and loving during these past three years and indeed our seven years together. I never had to worry if the stress of graduate school would break me. I always knew she would be there as she always has, believing in me, supporting me, holding me up in ways she probably doesn't even know. Thank you Christina, from the bottom of my heart.

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CHAPTER 1

INTRODUCTION

Using a mathematical model to represent some real-world phenomena requires patience, wisdom, and perhaps most of all knowledge. All such models have parameters associated with them. These parameters can be placed into two categories: input parameters and output parameters or performance measures. An input parameter is any number important to the model that the experimenter must provide. An example of an input parameter in a reliability model is the probability a system component functions; in a queueing model, an example of an input parameter is the rate at which customers arrive at the system. Performance measures are functions of the input parameters. The experimenter's goal is to know the value of a performance measure in the model. Knowledge about these parameters is always necessary in some capacity in order to use the model. In fact, this knowledge can be critical to the implementation of the model as the following example illustrates.

Suppose customers arrive at a bank that has only one teller. If the teller is idle, the arriving customer will be served, otherwise the customer will join a queue. Further assume the times between customer arrivals are independent exponential random variables with mean $\frac{1}{\lambda}$ and the service times are also independent exponential random variables with mean $\frac{1}{\mu}$. Consider using a stationary M/M/1 queue to model this system. In a stationary M/M/1 queue there are two input parameters, the arrival rate, λ and the service rate, μ. It is imperative to know the value of $\rho = \frac{\lambda}{\mu}$, the ratio of the arrival rate to the service rate. If this ratio is not less than one, the system is not stationary and a stationary model is therefore not appropriate. If the system is stationary and the

performance measure of interest is the mean waiting time in queue, W_q, then one can compute this output parameter, given the arrival rate and service rate:

$$W_q(\lambda, \mu) = \frac{\lambda}{\mu(\mu - \lambda)}.$$

As in the above example, if the model is stochastic and the input parameters are parameters of probability distributions in the model, then they are unknown and must be estimated from data. Even when using a deterministic model such as linear programming to solve an optimization problem, the coefficients involved could be unknown and would have to be estimated. The experimenter must decide how to best spend limited resources in trying to estimate the performance measure. For the M/M/1 queue, for example, should he allocate the resources equally in sampling from the arrival time distribution and the service time distribution, or should he concentrate his sampling efforts more heavily in one distribution or the other? This very important question is not limited to this specific example, but rather is pertinent anytime one wishes to estimate some performance measure of any model involving unknown parameters from more than one distribution.

Consider the problem of estimating a performance measure of a mathematical model with p input parameters, $\nu_1, \nu_2, \ldots, \nu_p$. Let the performance measure be denoted by $f(\nu_1, \nu_2, \ldots, \nu_p)$. If the performance measure is an unknown function of the input parameters, simulation must be used to determine the form of $f(\nu_1, \nu_2, \ldots, \nu_p)$. Even in the special case where the performance measure is a known function, the unknown parameters $\nu_1, \nu_2, \ldots, \nu_p$ in the model must be estimated in order to obtain an estimate $\widehat{f} = g(\widehat{\nu_1}, \widehat{\nu_2}, \ldots, \widehat{\nu_p})$ of the performance measure. This requires using a limited sampling budget to obtain the estimates $\widehat{\nu_1}, \widehat{\nu_2}, \ldots, \widehat{\nu_p}$. The experimenter's goal is that the estimator is "close" to the true performance measure. That is, he wants $f(\nu_1, \nu_2, \ldots, \nu_p) - \widehat{f}$ to be as close to zero as possible. Let $L(f, \widehat{f})$ be a loss function that measures how close the estimator, $\widehat{f}$, is to the true performance measure,

$f(\nu_1, \nu_2, \ldots, \nu_p)$. A reasonable and mathematically tractable loss function is the mean squared error (MSE) of $\widehat{f}$: $L(f, \widehat{f}) = E\left(\widehat{f} - f\right)^2$. Given some budgetary constraint, denoted b, the goal is to find the sample allocation that minimizes the MSE of the estimator. Let $V^0(b)$ be the MSE.

Faced with a mathematical model involving p parameters from q different distributions, $2 \leq q \leq p$, the problem can be stated as follows: Let X_{ij} $i = 1, 2, \ldots, q, j = 1, 2, \ldots, n_i$ be i.i.d. observations from population i, $c_1, c_2, \ldots, c_q$ be unit costs for sampling from populations $1, 2, \ldots, q$ respectively, $n_1, n_2, \ldots, n_q$ be the sample sizes for populations $1, 2, \ldots, q$ respectively, and $b > 0$ be a pre specified total sampling budget.

Find $n_1^{opt}, n_2^{opt}, \ldots, n_q^{opt}$ to:

$$\min E\left(\widehat{f} - f(\nu_1, \nu_2, \ldots, \nu_p)\right)^2$$
$$s.t. \sum_{i=1}^{q} c_i n_i \leq b$$

Assuming this optimization problem can be solved, the optimum set of sample sizes $\left(n_1^{opt}, n_2^{opt}, \ldots, n_q^{opt}\right)$ will be a function of the unknown parameters $\nu_1, \nu_2, \ldots, \nu_p$, and thus can not be computed without knowing $\nu_1, \nu_2, \ldots, \nu_p$. There are at least two approaches to solving this problem. One method uses a Bayesian approach in which the experimenter assumes a prior distributional form for each of the q distributions. A second approach involves a multiple stage sampling procedure. For example, a two stage sampling plan allocates a small portion of the sampling budget in stage 1 to obtain initial estimates of the unknown parameters. These estimates can then be used to compute final estimates of $\left(n_1^{opt}, n_2^{opt}, \ldots, n_q^{opt}\right)$, denoted $(n_1^{\star}, n_2^{\star}, \ldots, n_q^{\star})$. In stage 2, these estimated "optimal" sample sizes are used to collect the final samples. Hopefully, such a multiple stage sampling scheme would have the following optimality property: As the budget grows arbitrarily large, the MSE of the estimator produced by the multiple stage sampling scheme, denoted $V^{\star}(b)$, approaches the minimum MSE, $V^0(b)$, using the

optimal sample allocation. This multiple stage approach to determine a sampling allocation is useful even in the case where the experimenter does not know the value of his budget, b, exactly. Letting $t_i^{opt} = n_i^{opt}/b$, the experimenter can use $\left(t_1^{opt},\ t_2^{opt}, \ldots,\ t_q^{opt}\right)$ as a guide to allocate the undetermined budget.

CHAPTER 2

LITERATURE REVIEW

The problem of allocating a fixed budget among populations when estimating a function of parameters was originally addressed by Ghurye and Robbins (1959). Ghurye and Robbins considered the problem of allocating a fixed budget among two Normal populations when the function of interest is the difference in the two population means. That is, they solved the following problem: Let $f(\mu_1, \mu_2) = \mu_1 - \mu_2$. Find (n_1^{opt}, n_2^{opt}) such that the MSE of $\left(\overline{X}_{1n_1^{opt}} - \overline{X}_{2n_2^{opt}}\right)$ is minimized. Using the difference in the sample means as an estimator of their performance measure, they showed that the variance of $\left(\overline{X}_{1n_1^{opt}} - \overline{X}_{2n_2^{opt}}\right)$ is minimized by an allocation proportional to the population standard deviations. Since the population standard deviations are often unknown, they proposed a two-stage sampling plan in which they used the first stage to get an initial estimate of the population standard deviations. In the second stage, they allocated the remaining budget in an optimal fashion using estimates from the first stage. Their two-stage procedure follows:

<u>Stage One:</u>

(1) Start with initial random samples, $X_{11}, X_{12}, \ldots, X_{1m_1}, X_{21}, X_{22}, \ldots, X_{2m_2}$, of sizes m_1 and m_2, where $c_1 m_1 + c_2 m_2 < b$

(2) Compute sample estimates:

$$\widehat{\sigma}_i^2(m_i) = \frac{\sum_{j=1}^{m_i} X_{ij}^2 - m_i \overline{X}_{im_i}^2}{m_i - 1}, \; i = 1, 2$$

<u>Stage Two:</u>

(1) Compute $n_1^{\star} = \frac{c_1^{1/2}\widehat{\sigma}_1(m_1)}{\sum_{i=1}^{2} c_i^{1/2}\widehat{\sigma}_i(m_i)}$

$$N_1^\star = \begin{cases} m_1 & \text{if } \frac{b\, n_1^\star}{c_1} < m_1 \\ \frac{b - c_2 m_2}{c_1} & \text{if } \frac{b - c_2 m_2}{c_1} < \frac{b\, n_1^\star}{c_1} \\ \frac{b\, n_1^\star}{c_1} & \text{if } m_1 \leq \frac{b\, n_1^\star}{c_1} \leq \frac{b - c_2 m_2}{c_1} \end{cases}$$

$$N_2^\star = \frac{b - c_1 N_1^\star}{c_2}$$

$N_1 = \left[N_1^\star\right]$, $N_2 = \left[N_2^\star\right]$ where $[x]$ is the largest integer less than or equal to x.

(2) Sample $(N_1 - m_1)$ more observations from population 1,

Sample $(N_2 - m_2)$ more observations from population 2

(3) Estimate $f(\mu_1, \mu_2) = \mu_1 - \mu_2$ by $\overline{X}_{1N_1} - \overline{X}_{2N_2}$.

Ghurye and Robbins were able to show that their two stage sampling plan is optimal in the sense that the MSE of the two stage estimator approached the minimum MSE as the budget approached infinity. More specifically, they proved:

Theorem 2.1 (Ghurye and Robbins): *Let X_{ij}, $i = 1, 2$, $j = 1, 2, \ldots, N_i$ be i.i.d. observations from Normal(μ_i, σ_i^2) and $\widehat{\mu_i} = \overline{X}_i$. Let c_1, c_2, and $\rho = \frac{\sigma_2}{\sigma_1}$ remain fixed while m_1, m_2 and b become infinite in such a way that*

$$\begin{cases} 0 < h \leq \frac{m_1}{m_2} \leq h' < \infty, & \text{where } h,\ h' \text{ are fixed} \\ \frac{m_i}{b} \to 0 & i = 1, 2 \end{cases}$$

Then $V^\star(b)/V^0(b) \;\to\; 1$.

Ghurye and Robbins then compared their two-stage sampling scheme to the somewhat naive approach of allocating the budget equally among the Normal populations. Their results clearly showed the superiority of the two-stage plan for values of $\rho \geq 1.5$.

Ghurye and Robbins also showed the strict assumption of Normality is not necessary to their proof. Letting $F_i(\widehat{\sigma}; m) = \Pr\{\widehat{\sigma}_i(m) \leq \widehat{\sigma}\}$ and $A_i(\epsilon) = [\sigma_i - \epsilon,\ \sigma_i + \epsilon]$, one needs only assume:

(I) There exists an $\alpha > 1$ such that for every fixed $\epsilon > 0$,

$$m^{\alpha} \Pr\{\widehat{\sigma}_i(m) \notin A_i(\epsilon)\} \text{ is bounded for all } m > 0,\ i = 1, 2;$$

(II) There exists an $\epsilon > 0$ such that $\epsilon < \min(\sigma_1, \sigma_2)$ and

$$m \int_{A_i(\epsilon)} \widehat{\sigma}^k dF_i(\widehat{\sigma}; m) - \sigma_i^k \text{ is bounded for all } m > 0 \text{ and } k = 2,\ -2;$$

(III) Either $\overline{X}_i(m)$ and $\widehat{\sigma}_i(m)$ from the sample are a pair of mutually independent random variables, or each population has a finite fourth moment.

Finally, they showed that both Poisson and Binomial populations meet (I), (II), and (III).

Page (1990) attacked the problem of allocating a fixed budget among several populations when estimating the product of the means of the populations using the product of sample means. That is, $f(\nu_1, \nu_2, \ldots, \nu_p) = \prod_{i=1}^{p} \nu_i$ is the performance measure of interest and $g(\widehat{\nu}_1,\ \widehat{\nu}_2, \ldots, \widehat{\nu}_p) = \prod_{i=1}^{p} \widehat{\nu}_i$ where $\widehat{\nu}_i$ is the sample mean from population i. She showed that an allocation scheme based on the coefficients of variation for the populations minimizes a first order approximation to the variance of the product of the sample means estimator. Page used proportions of the budget rather than sample sizes to specify an allocation.-She first showed that an allocation exists that minimizes the variance of the estimator. More specifically she proved the following:

Theorem 2.2 (Page): *Fix b and let* $t = (t_1,\ t_2, \ldots,\ t_p)$ *where* $t_i = \frac{c_i n_i}{b}$ *and* $v_i = \frac{\sigma_i}{\mu_i}$ $i = 1, 2, \ldots, p$. *Then* $Var\left(\prod_{i=1}^{p} \widehat{\nu}_i\right)$ *is minimized by* $t = (t_1,\ t_2, \ldots,\ t_p)$ *with* $0 < t_i < 1$, *satisfying the following p equations:*

$$\frac{c_i v_i^2 t_i^{-2}}{c_i v_i^2 t_i^{-1} b^{-1} + 1} = \frac{c_p v_p^2 t_p^{-2}}{c_p v_p^2 t_p^{-1} b^{-1} + 1}\ i = 1, 2, \ldots, p-1,$$

$$\text{and } \sum_{i=1}^{p} t_i = 1.$$

She also proved that the coefficient of variation allocation, $t_i = \frac{c_i^{1/2} v_i}{\sum_j c_j^{1/2} v_j}$, $i = 1, 2, \ldots, p$ minimizes a first order approximation to $Var\left(\prod_{i=1}^{p} \widehat{\nu}_i\right)$.

Page also showed how to improve upon a balanced allocation scheme where each population is sampled equally. She compared different allocations by a measure called asymptotic relative efficiency (ARE). ARE(t, t') is interpreted as the limiting ratio of budgets needed to achieve equal variance of $\prod_{i=1}^{p} \widehat{\eta}_i$ with allocations t' and t, respectively. Though the coefficient of variation allocation is optimal, it is not useful in practice because the v_i are typically unknown. To deal with this problem, Page considered how partial or incomplete information, such as upper and lower bounds on the parameters, can be used to determine an allocation. She first proved that movement from balanced allocation in the direction of coefficient of variation allocation is an improvement. If cv-allocation is unknown, she provided a method of grouping the means based on incomplete information which yields upper and lower bounds on the cv-allocation. In another paper (Page 1985), she focused on the desirable Bayesian properties of such allocation schemes.

Zheng, Seila and Sriram (1997a) looked at the problem of estimating the product of p means with the product of sample means using a frequentist approach. They developed a two stage sampling procedure similar to that of Ghurye and Robbins and were able to prove the asymptotic optimality of their sampling scheme under some mild assumptions about the population distributions. Their two stage sampling plan for $p = 3$ populations follows:

<u>Stage One:</u>

(1) Start with initial random samples, $X_{11}, X_{12}, \ldots, X_{1m_0}$, $X_{21}, X_{22}, \ldots, X_{2m_0}$, and $X_{31}, X_{32}, \ldots, X_{3m_0}$, for a suitable m_0 defined below.

(2) Compute sample estimates:

$$\widehat{\mu}_1 = \overline{X}_{1m_0},\ \widehat{\mu}_2 = \overline{X}_{2m_0},\ \widehat{\mu}_3 = \overline{X}_{3m_0},$$

$$\widehat{\sigma}_i(m_0) = \frac{\sum_{j=1}^{m_0} (X_{ij} - \overline{X}_{im_0})^2}{m_0 - 1} \quad (i = 1, 2, 3)\,.$$

Stage Two:

(1) Let $n_1^\star = \frac{b\,\widehat{\mu}_2\widehat{\mu}_3\widehat{\sigma}_1}{\widehat{\mu}_2\widehat{\mu}_3\widehat{\sigma}_1 + \widehat{\mu}_1\widehat{\mu}_3\widehat{\sigma}_2 + \widehat{\mu}_1\widehat{\mu}_2\widehat{\sigma}_3}$,

$n_2^\star = \frac{b\widehat{\mu}_1\widehat{\mu}_3\widehat{\sigma}_2}{\widehat{\mu}_2\widehat{\mu}_3\widehat{\sigma}_1 + \widehat{\mu}_1\widehat{\mu}_3\widehat{\sigma}_2 + \widehat{\mu}_1\widehat{\mu}_2\widehat{\sigma}_3}$, and

$n_3^\star = \frac{b\widehat{\mu}_1\widehat{\mu}_2\widehat{\sigma}_3}{\widehat{\mu}_2\widehat{\mu}_3\widehat{\sigma}_1 + \widehat{\mu}_1\widehat{\mu}_3\widehat{\sigma}_2 + \widehat{\mu}_1\widehat{\mu}_2\widehat{\sigma}_3}$,

$$(2)\ N_1^\star = \begin{cases} m_0 & \text{if } n_1^\star \le m_0 \\ m_0 + \frac{n_1^\star(b-3m_0)}{n_1^\star + n_2^\star} & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star \le m_0 \\ m_0 + \frac{n_1^\star(b-3m_0)}{n_1^\star + n_3^\star} & \text{if } n_1^\star > m_0,\ n_2^\star \le m_0,\ n_3^\star > m_0 \\ b - 2m_0 & \text{if } n_1^\star > m_0,\ n_2^\star \le m_0,\ n_3^\star \le m_0 \\ n_1^\star & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \end{cases}$$

$$N_2^\star = \begin{cases} m_0 & \text{if } n_2^\star \le m_0 \\ m_0 + \frac{n_2^\star(b-3m_0)}{n_1^\star + n_2^\star} & \text{if } n_1^\star > m_0, n_2^\star > m_0,\ n_3^\star \le m_0 \\ m_0 + \frac{n^\star(b-3m_0)}{n_3^\star + n_2^\star} & \text{if } n_1^\star \le m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \\ b - 2m_0 & \text{if } n_1^\star \le m_0,\ n_2^\star > m_0, n_3^\star \le m_0 \\ n_2^\star & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \end{cases}$$

$$N_3^\star = b - N_1^\star - N_2^\star$$

and

(3) Compute the final sample sizes: $N_1 = [N_1^\star]$, $N_2 = [N_2^\star]$, $N_3 = [N_3^\star]$,

where $[x]$ is the largest integer less than or equal to x.

(4) Sample $(N_1 - m_0)$ more observations from population 1, $(N_2 - m_0)$ more observations from population 2, and $(N_3 - m_0)$ more observations from population 3.

(5) Estimate $\mu_1\mu_2\mu_3$ by $\overline{X}_{1N_1}\,\overline{X}_{2N_2}\,\overline{X}_{3N_3}$.

Specifically, in the case of $p = 3$ where $EX_i = \mu_i$ and $Var(X_i) = \sigma_i^2$, they proved the following:

Theorem 2.3 (Zheng, Seila, and Sriram): *Assume that $\mu_1,$ μ_2 and μ_3 are all positive. Define $D_i(\epsilon) = \left[\frac{\mu_i}{\sigma_i} - \epsilon, \frac{\mu_i}{\sigma_i} + \epsilon\right]$, $T_i(t) = \frac{\overline{X}_{it}}{\hat{\sigma}_i(t)}, i = 1, 2, 3.$ Assume populations 1, 2, and 3 satisfy:*

(1) $EX_1^4 < \infty$, $EX_2^4 < \infty$, and $EX_3^4 < \infty$;

(2) *There exists a $\beta > 1$ such that for every $\epsilon > 0$*

$$t^{\beta} P[T_i(t) \notin D_i(\epsilon)] = O(1), \text{ as } t \to \infty;$$

(3) *There exists an $\epsilon > 0$ and $\epsilon < min_{i=1,2,3}\left(\frac{\mu_i}{\sigma_i}\right)$ such that for each i = 1,2,3*

$$t\left\{\int_{[T_i(t)\in D_i(\epsilon)]} T_i^k(t)dP - \left(\frac{\mu_i}{\sigma_i}\right)^k\right\} = O(1), \text{ as } t \to \infty, \text{ for } k = -4, 4.$$

Then let the initial sample size $n_0 = b^{\alpha}$ where $\frac{4}{\beta+4} < \alpha < \frac{1 - ln\,(3)}{ln\,(b)}$ and β satisfies (2). For sample sizes $N_1, N_2,$ and N_3 defined by their two-stage sampling scheme and $h_0 = min\{2 - \alpha, \alpha(1 + \beta/2), 1 + \alpha\}$ we have:

$$E\left(\overline{X}_{1N_1}\,\overline{X}_{2N_2}\,\overline{X}_{3N_3} - \mu_1\mu_2\mu_3\right)^2 = V^0(b) + O\left(\tfrac{1}{b^{h_0}}\right) \text{ as } b \to \infty.$$

Consequently,

$$\frac{E\left(\overline{X}_{1N_1}\,\overline{X}_{2N_2}\,\overline{X}_{3N_3} - \mu_1\mu_2\mu_3\right)^2}{V^0(b)} \to 1 \text{ as } b \to \infty.$$

In addition to proving the asymptotic optimality of their sampling scheme, they performed simulations assuming Normal distributions for the $p = 3$ case. Their experiments demonstrated the superiority of the two-stage sampling scheme over a simple one-stage approach which evenly allocates the sampling budget among the three populations. They also demonstrated the optimality of the two-stage scheme by showing that the variance of the estimator in their sampling plan approaches $V^0(b)$ as b gets larger. Additionally, they tested the sensitivity of α, and thus initial sample size, by running simulations with different values of α. The results showed that different initial sample sizes do affect the results, but this effect is diminished as b grows larger. In a previous paper, Zheng, Seila and Sriram (1996) showed that Normal, Poisson, Exponential and Bernoulli populations meet their mild assumptions.

Zheng, Seila and Sriram (1997b) broadened the range of models by studying the problem of allocating a fixed budget among the arrival and service time distributions of an M/M/1 queue. Zheng and Seila (1996) showed that the substitution estimator for mean waiting time in queue, $\widehat{W}_{1q} = \frac{\overline{X}^2_{2n_2}}{\overline{X}_{1n_1} - \overline{X}_{2n_2}}$, which is obtained by substituting $\frac{1}{\overline{X}_{1n_1}}$ for λ and $\frac{1}{\overline{X}_{2n_2}}$ for μ into the expression for mean waiting time, $W_q(\lambda, \mu) = \frac{\lambda}{\mu(\mu-\lambda)}$, has infinite mean squared error. This result was already known; see Schruben and Kulkarni (1982).

Theorem 2.4 (Zheng and Seila): *By assuming $\rho = \frac{\lambda}{\mu} < \rho_0 < 1$, where ρ_0 is known, the alternative estimator*

$$\widehat{W}_q = \begin{cases} \frac{\overline{X}^2_{2n_2}}{\overline{X}_{1n_1} - \overline{X}_{2n_2}} & \text{if } \overline{X}_{2n_2} \leq \rho_0 \, \overline{X}_{1n_1} \\ \frac{\rho_0}{1-\rho_0} \overline{X}_{2n_2} & \text{otherwise} \end{cases}$$

has the following properties:

$$\widehat{W}_q \xrightarrow{\text{a.s.}} W_q \text{ as } n_1, n_2 \to \infty,$$

$$E\widehat{W}_q = W_q + O(\tfrac{1}{n_1}) + O(\tfrac{1}{n_2}), \text{ and}$$

$$E\left(\widehat{W}_q - W_q\right)^2 = \frac{\lambda^2}{(\mu-\lambda)^4}\frac{1}{n_1} + \frac{(2\lambda\mu-\lambda^2)^2}{\mu^2(\mu-\lambda)^4}\frac{1}{n_2} + O(\tfrac{1}{n_1^2}) + O(\tfrac{1}{n_2^2}) + O(\tfrac{1}{n_1 n_2}).$$

Using this alternative estimator, Zheng, Seila, and Sriram developed the following two stage sampling scheme:

Stage One:

(1) Start with initial random samples, $X_{11}, X_{12}, \ldots, X_{1n_0}$ of interarrival times, and $X_{21}, X_{22}, \ldots, X_{2n_0}$ of service times, for a suitable n_0 defined below.

(2) Compute sample estimates:

$$\widehat{\lambda} = \frac{1}{\bar{X}_{1n_0}} \text{ and } \widehat{\mu} = \frac{1}{\bar{X}_{2n_0}}$$

Stage Two:

(1) Let
$$n_1^\star = \frac{b\,\widehat{\mu}}{c_1\widehat{\mu} + c_1^{1/2}c_2^{1/2}2\left(\widehat{\mu}-\widehat{\lambda}\right)},$$
$$n_2^\star = \frac{b2\left(\widehat{\mu}-\widehat{\lambda}\right)}{c_1\widehat{\mu} + c_1^{1/2}c_2^{1/2}2\left(\widehat{\mu}-\widehat{\lambda}\right)},$$

(2)
$$N_1^\star = \begin{cases} n_0 & \text{if } n_1^\star \le n_0 \\ \frac{b-c_2n_0}{c_1} & \text{if } n_1^\star \ge \frac{b-c_2n_0}{c_1} \\ n_1^\star & \text{if } n_0 < n_1^\star < \frac{b-c_2n_0}{c_1} \end{cases}$$

$$N_2^\star = \frac{b-c_1N_1^\star}{c_2}$$

(3) Compute the final sample sizes: $N_1 = [N_1^\star]$, $N_2 = [N_2^\star]$ where $[x]$ is the largest integer less than or equal to x.

(4) Take $(N_1 - n_0)$ more interarrival time observations and $(N_2 - n_0)$ more service time observations.

Specifically, they proved the following optimality property for this estimator:

Theorem 2.5 (Zheng, Seila, and Sriram): *Let* $\alpha \in \left(0.5, 1 - \frac{ln(c_1+c_2)}{ln(b)}\right)$ *and* $h_0 = min(2-\alpha, 1+\alpha)$. *Then for their two-stage procedure and initial sample size* $n_0 = b^\alpha$:

(1) $E(\widehat{W}_q - W_q)^2 = V^0(b) + O(\frac{1}{b^{h_0}})$,

(2) *The optimal value of* α *is between 0.5 and* $1 - \frac{ln(c_1+c_2)}{ln(b)}$, *and* $h_0 \geq 1.5$, *for any* $\alpha \in \left(0.5, 1 - \frac{\ln(c_1+c_2)}{\ln(b)}\right)$.

Zheng, Seila, and Sriram ran simulations to support their theoretical results. Their empirical work clearly showed the MSE of their two stage estimator tends to the minimum variance as the sample size gets larger and it only takes relatively small sample sizes to get reasonably close. Additionally, their simulations showed that the rate of convergence slows as ρ, the traffic intensity, increases.

The results were extended to three other system performance measures of the M/M/1 queue: mean waiting time in system, mean number of customers in the system, and mean number of customers in the queue. The results were also broadened to include the M/E_k/1 queue.

CHAPTER 3

A RELIABILITY MODEL

We will consider the problem of allocating a sample to estimate the following function of three population means by the corresponding function of sample means:

$$f(\mu_1, \mu_2, \mu_3) = \mu_1(\mu_2 + \mu_3) \tag{3.1}$$

The problem is to allocate a fixed sampling budget to the three populations with a goal of minimizing the MSE of the estimator. Let $X_{i1}, X_{i2}, \ldots$ be i.i.d. observations from population i with unknown mean μ_i and variance σ_i^2, $i = 1, 2, 3$. Assume observations from the three populations are mutually independent. We wish to determine optimal sample sizes $(n_1^{opt}, n_2^{opt}, n_3^{opt})$ which minimize the first order approximation of the MSE

$$V_{n_1,n_2,n_3} = E(\overline{X}_{1n_1}(\overline{X}_{2n_2} + \overline{X}_{3n_3}) - \mu_1(\mu_2 + \mu_3))^2, \tag{3.2}$$

subject to the constraint that the total sampling cost is $c_1 n_1 + c_2 n_2 + c_3 n_3 \leq b$, where c_i is the unit cost for sampling the i-th population, n_i is the sample size for the ith populations and b is a pre specified total sampling budget.

3.1 First Order Allocation

Using a Taylor series expansion, one can show that

$$V_{n_1,n_2,n_3} = \widetilde{V}_{n_1,n_2,n_3} + O((n_1 n_2)^{-1}) + O((n_1 n_3)^{-1}), \tag{3.3}$$

where

$$\widetilde{V}_{n_1,n_2,n_3} = (\mu_2 + \mu_3)^2 \frac{\sigma_1^2}{n_1} + \mu_1^2 \frac{\sigma_2^2}{n_2} + \mu_1^2 \frac{\sigma_3^2}{n_3} \tag{3.4}$$

is the first order approximation. The values $(n_1^{opt}, n_2^{opt}, n_3^{opt})$, referred to as the *first-order allocation*, which minimize $\widetilde{V}_{n_1,n_2,n_3}$ are

$$\begin{cases} n_1^{opt} = & \frac{b\,\sigma_1(\mu_2+\mu_3)}{c_1\,\sigma_1(\mu_2+\mu_3)+c_2\mu_1\sigma_2+c_3\mu_1\sigma_3} \\ n_2^{opt} = & \frac{b\mu_1\sigma_2}{c_1\,\sigma_1(\mu_2+\mu_3)+c_2\mu_1\sigma_2+c_3\mu_1\sigma_3} \\ n_3^{opt} = & \frac{b\mu_1\sigma_3}{c_1\,\sigma_1(\mu_2+\mu_3)+c_2\mu_1\sigma_2+c_3\mu_1\sigma_3} \end{cases} \tag{3.5}$$

Therefore, one can show that the lower bound for the first order approximation of the MSE is given by

$$V_{low} = V_{n_1^{opt}, n_2^{opt}, n_3^{opt}} = V^0(b) + O(\frac{1}{b^2}), \tag{3.6}$$

with

$$V^0(b) = \widetilde{V}_{n_1^{opt}, n_2^{opt}, n_3^{opt}}. \tag{3.7}$$

Because the first-order allocation $(n_1^{opt}, n_2^{opt}, n_3^{opt})$ depends on the unknown parameters μ_1, μ_2, μ_3, σ_1, σ_2, and σ_3, we will propose a two-stage sampling procedure similar to that in Zheng, Seila, and Sriram (1997a) and establish its optimality properties as the total budget goes to infinity.

3.2 Two Stage Sampling Procedure

The objective of the two-stage procedure is to determine the final sample sizes that minimize the first-order approximation of the MSE in (3.4) subject to the fixed overall budget (sample size). The procedure works as follows: In stage 1 we select equal-sized samples from each population. Then we estimate n_1^{opt}, n_2^{opt}, and n_3^{opt} in (3.5) by $n_1^\star$, $n_2^\star$, and $n_3^\star$ using sample estimates of μ_1, μ_2, μ_3, σ_1, σ_2, and σ_3 computed from the initial sample. In stage 2, we allocate the remaining budget based on the stage 1 results. The detailed procedure follows:

Stage One:

(1) Start with initial random samples $X_{11}, X_{12}, ..., X_{1m_0}$, $X_{21}, X_{22}, \ldots, X_{2m_0}$, and $X_{31}, X_{32}, \ldots, X_{3m_0}$, for a suitable m_0 to be defined below.

(2) Compute sample estimates:

$$\widehat{\mu}_i = \overline{X}_{im_0},\ i = 1, 2, 3,$$
$$\widehat{\sigma}_i(m_0) = \frac{\sum_{j=1}^{m_0} (X_{ij} - \overline{X}_{im_0})^2}{m_0 - 1} \quad (i = 1, 2, 3)\,.$$

(3) Let
$$\begin{aligned} n_1^\star &= \frac{b\,\widehat{\sigma}_1(\widehat{\mu}_2+\widehat{\mu}_3)}{c_1\,\widehat{\sigma}_1(\widehat{\mu}_2+\widehat{\mu}_3)+c_2\widehat{\mu}_1\widehat{\sigma}_2+c_3\widehat{\mu}_1\widehat{\sigma}_3}, \\ n_2^\star &= \frac{b\widehat{\mu}_1\widehat{\sigma}_2}{c_1\,\widehat{\sigma}_1(\widehat{\mu}_2+\widehat{\mu}_3)+c_2\widehat{\mu}_1\widehat{\sigma}_2+c_3\widehat{\mu}_1\widehat{\sigma}_3}, \\ n_3^\star &= \frac{b\widehat{\mu}_1\widehat{\sigma}_3}{c_1\,\widehat{\sigma}_1(\widehat{\mu}_2+\widehat{\mu}_3)+c_2\widehat{\mu}_1\widehat{\sigma}_2+c_3\widehat{\mu}_1\widehat{\sigma}_3}. \end{aligned} \tag{3.8}$$

Stage Two:

(1) Let

$$N_1^\star = \begin{cases} m_0 & \text{if } n_1^\star \le m_0 \\ m_0 + \frac{n_1^\star(b-(c_1+c_2+c_3)m_0)}{(c_1n_1^\star+c_2n_2^\star)} & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star \le m_0 \\ m_0 + \frac{n_1^\star(b-(c_1+c_2+c_3)m_0)}{(c_1n_1^\star+c_3n_3^\star)} & \text{if } n_1^\star > m_0,\ n_2^\star \le m_0,\ n_3^\star > m_0 \\ (b-(c_2+c_3)m_0)/c_1 & \text{if } n_1^\star > m_0,\ n_2^\star \le m_0,\ n_3^\star \le m_0 \\ n_1^\star & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \end{cases} \tag{3.9}$$

$$N_2^\star = \begin{cases} m_0 & \text{if } n_2^\star \le m_0 \\ m_0 + \frac{n_2^\star(b-(c_1+c_2+c_3)m_0)}{(c_2n_2^\star+c_1n_1^\star)} & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star \le m_0 \\ m_0 + \frac{n_2^\star(b-(c_1+c_2+c_3)m_0)}{(c_2n_2^\star+c_3n_3^\star)} & \text{if } n_1^\star \le m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \\ (b-(c_1+c_3)m_0)/c_2 & \text{if } n_1^\star \le m_0,\ n_2^\star > m_0,\ n_3^\star \le m_0 \\ n_2^\star & \text{if } n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star > m_0 \end{cases} \tag{3.10}$$

$$N_3^\star = (b - \ c_1N_1^\star - \ c_2N_2^\star)/c_3 \tag{3.11}$$

and

$$N_1 = [N_1^\star],\ N_2 = [N_2^\star],\ \ N_3 = [b - N_1 - N_2], \tag{3.12}$$

where $[x]$ is the largest integer less than or equal to x.

(2) Sample $(N_i - m_0)$ more observations from population i, $i = 1, 2, 3$.

(3) Finally, estimate $\mu_1(\mu_2 + \mu_3)$ by $\overline{X}_{1N_1}(\overline{X}_{2N_2} + \overline{X}_{3N_3})$. (3.13)

In step (1) of stage two, the estimated optimal values $n_1^\star$, $n_2^\star$, and $n_3^\star$ computed using (3.8) may give rise to the following three cases: (i) only one of them, say $n_1^\star$, is greater than m_0; (ii) exactly two of them, say $n_1^\star$and $n_2^\star$, are greater than m_0; and (iii) all three of them are greater than m_0. Note that all these possibilities are identified in sets A_1 to A_7 in Chapter 6 of this thesis. In case (i), all additional observations are sampled from population 1 since the stage 1 sample sizes for populations 2 and 3 are larger than the estimated optimal sample sizes for these populations. In case (ii), all additional observations are sampled proportionally from populations 1 and 2 since the stage 1 sample size for population 3 is larger than the estimated optimal sample size for this population. Finally, in case (iii), take $n_1^\star$, $n_2^\star$, and $n_3^\star$ to be the final sample sizes since all of them exceed the stage 1 sample sizes.

3.3 Optimal Properties of the Two Stage Procedure

Theorem 3.1 below shows that the two stage procedure defined in (3.8) to (3.12) is asymptotically risk efficient. That is, as the budget grows arbitrarily large, the MSE of the estimator in (3.13) approaches the minimum MSE, $V^0(b)$ from (3.7). Before we state the theorem, we give a list of sufficient conditions needed to prove the theorem. Assume that μ_1, μ_2 and μ_3 are all positive. Let I_A denote the indicator function of a set A and for $\epsilon > 0$, define

$$D_1(\epsilon) = \left[\tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} - \epsilon, \tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} + \epsilon\right] \tag{3.14}$$

$$D_2(\epsilon) = \left[\tfrac{\sigma_3}{\sigma_2} - \epsilon, \tfrac{\sigma_3}{\sigma_2} + \epsilon\right] \tag{3.15}$$

$$D_3(\epsilon) = \left[\tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} - \epsilon, \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} + \epsilon\right] \tag{3.16}$$

$$D_4(\epsilon) = \left[\frac{\sigma_2}{\sigma_3} - \epsilon, \frac{\sigma_2}{\sigma_3} + \epsilon\right] \tag{3.17}$$

$$D_5(\epsilon) = \left[\frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} - \epsilon, \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} + \epsilon\right] \tag{3.18}$$

$$R_1 = \frac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} \tag{3.19}$$

$$R_2 = \frac{\sigma_3}{\sigma_2} \tag{3.20}$$

$$R_3 = \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} \tag{3.21}$$

$$R_4 = \frac{\sigma_2}{\sigma_3} \tag{3.22}$$

$$R_5 = \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} \tag{3.23}$$

$$T_1(n) = \frac{\overline{X}_{1n}(S_{2n}+S_{3n})}{S_{1n}(\overline{X}_{2n}+\overline{X}_{3n})} \tag{3.24}$$

$$T_2(n) = \frac{S_{3n}}{S_{2n}} \tag{3.25}$$

$$T_3(n) = \frac{S_{1n}(\overline{X}_{2n}+\overline{X}_{3n})}{\overline{X}_{1n}S_{2n}} \tag{3.26}$$

$$T_4(n) = \frac{S_{2n}}{S_{3n}} \tag{3.27}$$

$$T_5(n) = \frac{S_{1n}(\overline{X}_{2n}+\overline{X}_{3n})}{\overline{X}_{1n}S_{3n}} \tag{3.28}$$

The sufficient conditions are:

(1) $EX_i^4 < \infty, i = 1, 2, 3;$ (3.29)

(2) There exists a $\beta > 1$ such that for every $\epsilon > 0$

$$t^\beta P[T_i(t) \notin D_i(\epsilon)] = O(1),\ i = 1, 2, 3, 4, 5,\ \text{as } t \to \infty; \tag{3.30}$$

(3) There exists an $\epsilon > 0$ and $\epsilon < \min_{i=1,2,3,4,5} R_i$ such that for each $i = 1, 2, 3, 4, 5$

$$t\{\int_{[T_i(t)\in D_i(\epsilon)]} T_i^k(t)dP - R_i^k\} = O(1),\ \text{as } t \to \infty;\ \text{for } k = -4, 4. \tag{3.31}$$

Theorem 3.1 (Risk Efficiency): *Suppose that populations 1, 2 and 3 satisfy conditions* (3.29) to (3.31). *For a β satisfying* (3.30), *let* $\frac{4}{\beta+4} < \alpha < 1 - \frac{ln(3)}{ln(b)}$ *and assume that the initial sample size* $m_0 = b^\alpha$. *For N_i defined in* (3.12) *and* $h_0 = min\{2-\alpha, \alpha(1+\frac{\beta}{2}), 1+\alpha\}$ *we have*

$$E(\overline{X}_{1N_1}(\overline{X}_{2N_2} + \overline{X}_{3N_3}) - \mu_1(\mu_2+\mu_3))^2 = V^0(b) + O(\tfrac{1}{b^{h_0}}), \text{ as } b \to \infty, \tag{3.32}$$

where $V^0(b)$ *is defined as in* (3.7).

Proof: See Chapter 6.

It can be shown that Normal, Poisson, Exponential, and Bernoulli populations satisfy the conditions in (3.29) to (3.31) for any $\beta > 1$. For example see Zheng, Seila, and Sriram (1995). For discrete populations, such as Poisson and Bernoulli, one needs to modify the two stage procedure slightly to account for the fact that the sample means and variances could be zero with positive probability. One need only use the following modified estimators.

$$\widehat{\mu}_i = \begin{cases} \frac{1}{m_0} & \text{if } \overline{X}_{im_0} = 0 \\ \overline{X}_{im_0} & \text{otherwise} \end{cases} \tag{3.33}$$

$$\widehat{\sigma}_i^2 = \begin{cases} \frac{1}{m_0} & \text{if } s_i^2 = 0 \\ s_i^2 & \text{otherwise} \end{cases}$$

The estimators in (3.33) were used in the simulations described in the next section. This modification is important because if $c_1\widehat{\sigma}_1(\widehat{\mu}_2 + \widehat{\mu}_3) + c_2\widehat{\mu}_1\widehat{\sigma}_2 + c_3\widehat{\mu}_1\widehat{\sigma}_3 = 0$, $n_i^\star$ are undefined. The modification also solves a more subtle difficulty. For example, if $\mu_1 = 0.96$, $\mu_2 = 0.5$, $\mu_3 = 0.99$, $b = 800$, and the populations are binomial, then the optimal allocation is (269.26, 442.65, 88.09). If a failure does not occur in population 3 in the initial sample, $s_3^2 = 0$ and therefore $n_3^\star = 0$, a poor estimate of n_3^{opt} indeed.

3.4 Empirical Results

Suppose we want to estimate the function in (3.1) where the unknown means are from three Bernoulli populations. In this situation, a one stage sampling scheme might divide

the given total sampling budget b as follows: sample $[\frac{b}{2}]$ observations from population 1, and $[\frac{b}{4}]$ observations each from populations 2 and 3. Then the one stage estimator of $\mu_1(\mu_2+\mu_3)$ is $\overline{X}_{1[\frac{b}{2}]}(\overline{X}_{2[\frac{b}{4}]}+\overline{X}_{3[\frac{b}{4}]})$. We present some simulation results that demonstrate the performance of our two stage estimator (based on the two stage sampling scheme) is better than the one stage estimator. Furthermore, we demonstrate the asymptotic risk efficiency of our two stage estimator.

Note that the MSE of the one stage estimator if $\frac{b}{4}$ is an integer is

$$V_{one} = 2(\mu_2+\mu_3)^2\frac{\sigma_1^2}{b} + 4\mu_1^2\frac{\sigma_2^2}{b} + 4\mu_1^2\frac{\sigma_3^2}{b} + O(\frac{1}{b^2}), \text{ as } b \to \infty.$$

Let V_{two} denote the MSE of the two stage procedure (see 3.32) and let V_{low} be as defined in (3.6). For the simulations we assume Bernoulli populations with $\mu_1 = 0.99$, $\mu_2 = 0.45$, and $\mu_3 = 0.3$. We use equal sampling cost for each population, that is $c_i = 1$ for $i = 1, 2, 3$. The simulation was carried out using a 31-bit prime modulus multiplicative congruential random number generator with modulus $2^{31}-1$. The random number generator uses the multiplier 742938285. Turbo Pascal was the programming language used.

Let $\widehat{V}_{one}$ be the estimator of V_{one} and $\widehat{V}_{two}$ be the estimator of V_{two}, based upon n replications. Table 3.1 gives values for $\widehat{V}_{one}/V_{low}$ and values for $\widehat{V}_{two}/V_{low}$ using $\alpha = 0.5$ along with its 95% confidence intervals for different values of b. We used 10,000 iterations to get these values. The limiting value of V_{one}/V_{low} is 1.732. From this table we see that $\widehat{V}_{two}/V_{low}$ is smaller than V_{one}/V_{low} thus demonstrating the superiority of the two stage sampling procedure over the one stage plan. We also see from the table that $\widehat{V}_{two}/V_{low}$ becomes closer and closer to one as b increases. This shows the MSE of our two stage estimator approaches the theoretical minimum MSE as $b \to \infty$, providing

empirical evidence for the results proved in Theorem 3.1. Figure 3.1 gives the graphs for the results in Table 3.1.

Table 3.1: Optimality of Two Stage Estimator with 10,000 Iterations

	b						
	20	40	80	100	200	400	800
$\widehat{V}_{one}/V_{low}$	1.691	1.748	1.726	1.715	1.754	1.744	1.706
$\widehat{V}_{two}/V_{low}$	1.438	1.373	1.206	1.176	1.145	1.061	1.027
95% CI Ubd	1.479	1.412	1.240	1.208	1.176	1.091	1.055
95% CI Lbd	1.398	1.324	1.170	1.143	1.113	1.032	1.000

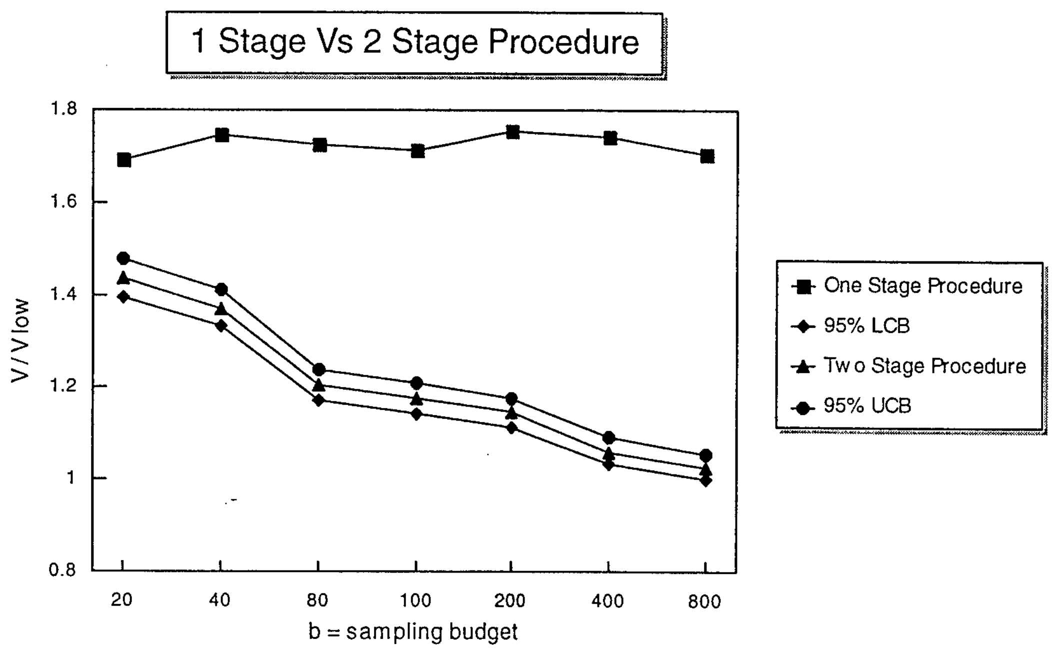

Figure 3.1

We also conducted simulations to test the effect of different values of α and thus different initial sample sizes. Table 3.2 gives $\widehat{V}_{two}/V_{low}$ and its 95% confidence intervals for different values of α and b; figure 3.2 displays this information graphically.

Table 3.2 Estimated V_{two}/V_{low} and Its 95% CI with 10,000 Iterations

$\mu_1 = 0.99,\ \mu_2 = 0.45,\ \mu_3 = 0.3$				
b	α	95% Lbd	$\widehat{V}_{two}/V^0(b)$	95% Ubd
20	0.4	1.471	1.515	1.558
	0.5	1.398	1.438	1.479
	0.6	1.340	1.375	1.410
40	0.4	1.325	1.363	1.401
	0.5	1.334	1.373	1.412
	0.6	1.445	1.483	1.521
80	0.4	1.222	1.257	1.293
	0.5	1.192	1.206	1.240
	0.6	1.320	1.355	1.390
100	0.4	1.209	1.243	1.278
	0.5	1.143	1.176	1.208
	0.6	1.248	1.281	1.315
200	0.4	1.166	1.199	1.232
	0.5	1.113	1.145	1.176
	0.6	1.119	1.151	1.183
400	0.4	1.093	1.124	1.156
	0.5	1.032	1.061	1.091
	0.6	0.992	1.020	1.048
800	0.4	1.053	1.083	1.113
	0.5	1.000	1.027	1.055
	0.6	0.968	0.993	1.019

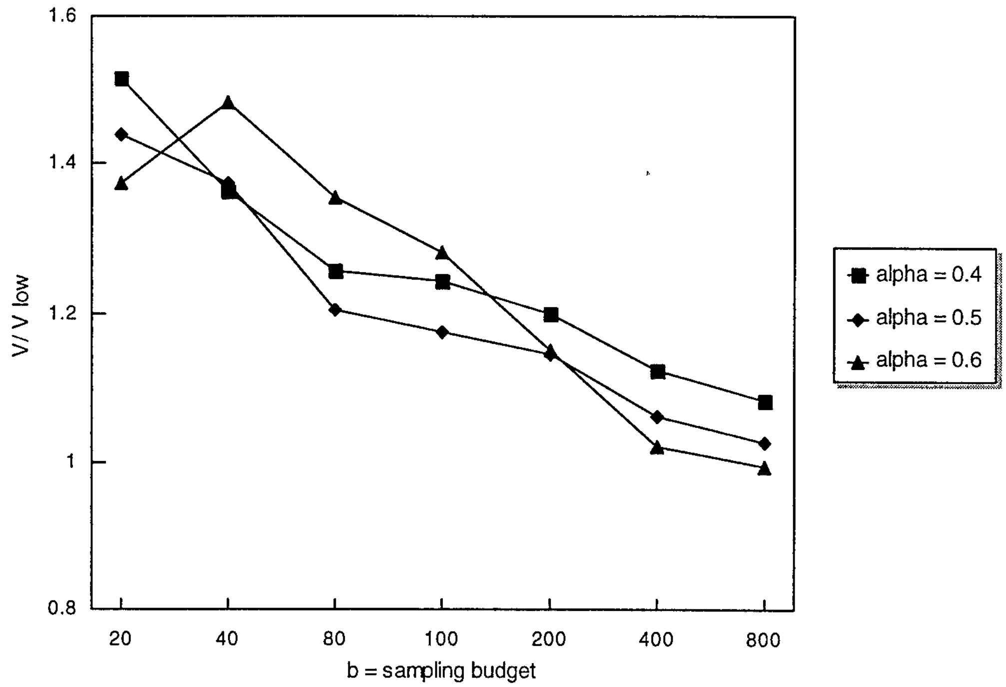

Figure 3.2

Table 3.2 and Figure 3.2 indicate there is no clear preference among the different values for α for all budget sizes; however, the smaller values of α seem to perform better at small values of b and $\alpha = 0.6$ seems to be a better choice for larger values of b.

Table 3.3 gives $\widehat{V}_{two}/V_{low}$ and its 95% confidence intervals for different values of α and b for a different set of population means. For these simulation results, $\mu_1 = 0.99$, $\mu_2 = 0.95$, and $\mu_3 = 0.9$. The value for α labeled "variable" is determined by the function $\alpha = (0.5 + 1 - \ln(3)/\ln(b))/2$. This allows the value of α to change along with b. Figure 3.3 displays the information in Table 3.3 graphically.

Table 3.3 Estimated V_{two}/V_{low} and Its 95% CI with 10,000 Iterations

$\mu_1 = 0.99,\ \mu_2 = 0.95,\ \mu_3 = 0.9$				
b	α	95% Lbd	$\widehat{V}_{two}/V^0(b)$	95% Ubd
40	0.5	1.064	1.095	1.125
	0.6	1.253	1.285	1.479
	variable	1.252	1.284	1.316
80	0.5	1.099	1.131	1.162
	0.6	1.038	1.066	1.093
	variable	0.998	1.025	1.052
100	0.5	1.079	1.109	1.139
	0.6	1.072	1.102	1.131
	variable	1.021	1.049	1.078
200	0.5	1.112	1.144	1.175
	0.6	1.060	1.088	1.117
	variable	1.038	1.067	1.096
400	0.5	1.108	1.139	1.171
	0.6	1.074	1.104	1.134
	variable	1.059	1.089	1.119
800	0.5	1.089	1.119	1.150
	0.6	1.016	1.045	1.073
	variable	1.008	1.037	1.066
1000	0.5	1.092	1.122	1.153
	0.6	1.064	1.093	1.123
	variable	0.996	1.025	1.053
2000	0.5	1.070	1.100	1.130
	0.6	1.000	1.029	1.058
	variable	0.990	1.019	1.047
4000	0.5	1.037	1.066	1.095
	0.6	0.992	1.021	1.049
	variable	1.013	1.042	1.071
8000	0.5	1.003	1.031	1.061
	0.6	0.984	1.012	1.041
	variable	1.011	1.040	1.069

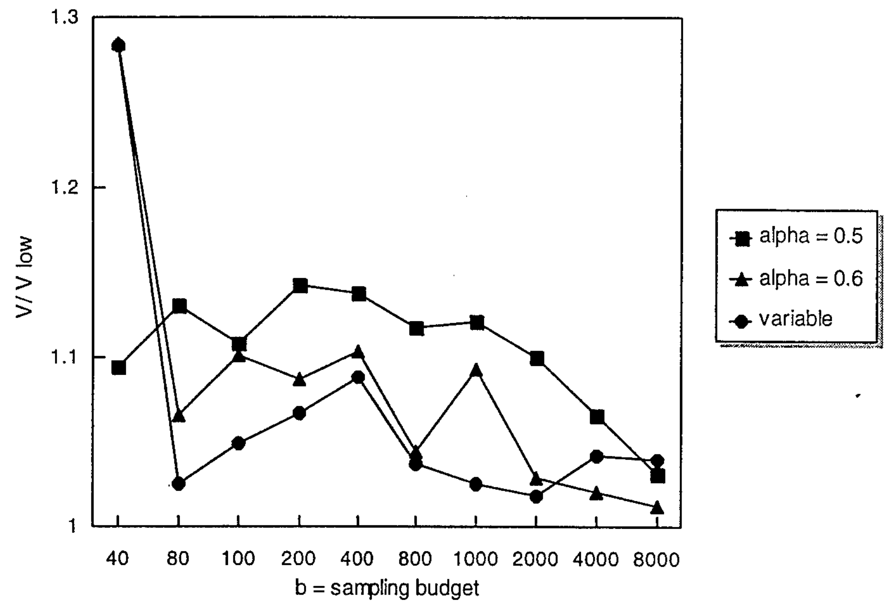

Figure 3.3

Table 3.3 and Figure 3.3 indicate $\alpha = 0.5$ is the best choice for small values of b while using a "variable" value for α is the best choice for values of b between 80 and 2000. For larger values of b, $\alpha = 0.6$ seems to be the best choice. Table 3.3 and Figure 3.3 also indicate the convergence of $\widehat{V}_{two}$ to V_{low} is slower when the population means are close to 1.

CHAPTER 4

THE M/G/1 QUEUEING MODEL

We will consider the problem of optimally allocating a sampling budget between the interarrival times and service times for the stationary M/G/1 queue with the goal of minimizing the mean squared error of an estimator of the mean waiting time in queue. Let $X_{11}, X_{12}, \ldots$ be i.i.d. observations from the interarrival time distribution with mean β and $X_{21}, X_{22}, \ldots$ be i.i.d. observations from the service time distribution with mean τ and variance σ^2. Assume observations from the two distributions are mutually independent. Additionally assume the traffic intensity $\rho = \frac{\tau}{\beta} < 1$. Since $\rho < 1$, the system is stationary and the mean waiting time in queue can be computed. This is the performance measure we will estimate:

$$W_q(\beta, \tau, \sigma^2) = \frac{\sigma^2 + \tau^2}{2(\beta - \tau)}. \tag{4.1}$$

(4.1) is taken from expression (8.33) in section 8.5 of Ross (1993) with $\frac{1}{\lambda} = \beta$.

4.1 Properties of Estimators of Mean Waiting Time

In the case of the M/M/1 queue, Zheng, Seila, and Sriram (1997b) showed that the substitution estimator for the mean waiting time in queue has infinite MSE. The problem of estimating mean waiting time in the M/G/1 queue is different from that in the M/M/1 queue because in the M/M/1 queue the service time distribution has only one parameter, the mean, to be estimated. In the M/G/1 queue, we will need to estimate two parameters from the service time distribution, the mean and variance: τ and σ^2. Still, the natural estimator of the mean waiting time, $W_q\left(\widehat{\beta}, \widehat{\tau}, \widehat{\sigma}^2\right)$, obtained by substituting $\overline{X}_{1n_1}$ for the mean interarrival time, $\overline{X}_{2n_2}$ for the mean service time, and $s^2 = \frac{\sum_{i=1}^{n_2}\left(X_i - \overline{X}_{2n_2}\right)^2}{n_2 - 1}$ for the

variance of the service times, has infinite mean squared error. The following theorem establishes this fact. The proof is given in Chapter 6.

Theorem 4.1: *Let* $\widehat{W}_{1q} = \frac{s^2+\overline{x}_{2n_2}^2}{2(\overline{x}_{1n_1}-\overline{x}_{2n_2})}$. *Then the following hold:*

$$E\left|\widehat{W}_{1q}\right| = +\infty, \tag{4.2}$$

$$E(\widehat{W}_{1q} - W_q)^2 = +\infty. \tag{4.3}$$

Since $\widehat{W}_{1q}$ has infinite mean squared error, it is not useful to estimate mean waiting time in the queue, especially since our objective is to find the sample allocation with minimum mean squared error. We propose the following alternative estimator: Let $\rho_0 < 1$ and assume that $\rho < \rho_0$. Define

$$\widehat{W}_q = \begin{cases} \frac{s^2+\overline{X}_{2n_2}^2}{2(\overline{X}_{1n_1}-\overline{X}_{2n_2})} & \text{if } \overline{X}_{2n_2}, \leq \rho_0 \overline{X}_{1n_1}, \\ \frac{\rho_0 \overline{X}_{2n_2}\left(\frac{s^2}{\overline{X}_{2n_2}^2}+1\right)}{2(1-\rho_0)} & \text{otherwise} \end{cases} \tag{4.4}$$

The following theorem establishes the statistical properties of the alternative estimator, $\widehat{W}_q$ as defined in (4.4). The proof is given in Chapter 6.

Theorem 4.2: *For the estimator defined by* (4.4) *in an M/G/1 queue with traffic intensity* $\rho < \rho_0 < 1$, *where* ρ_0 *is known,*

$$E\widehat{W}_q = W_q + O(\frac{1}{n_1}) + O(\frac{1}{n_2}), \tag{4.5}$$

$$\begin{aligned} E(\widehat{W}_q - W_q)^2 = {} & \frac{(\sigma^2+\tau)^2}{4(\beta-\tau)^4}\frac{\beta^2}{n_1} + \frac{(2\beta\tau-\tau^2+\sigma^2)^2}{4(\beta-\tau)^4}\frac{\sigma^2}{n_2} + \frac{1}{4(\beta-\tau)^2}\left(\frac{\kappa^4-\sigma^4}{n_2}\right) \\ & + \frac{2\beta\tau-\tau^2+\sigma^2}{2(\beta-\tau)^3}\frac{\kappa^3}{n_2} + O(\frac{1}{n_1^2}) + O(\frac{1}{n_2^2}) + O(\frac{1}{n_1 n_2}), \end{aligned} \tag{4.6}$$

where κ^3 and κ^4 are the third and fourth central moments respectively of the service time distribution.

4.2 First Order Allocation

Therefore, using $\widehat{W}_q$ to estimate the mean waiting time, our goal is to determine optimal sample sizes (n_1^{opt}, n_2^{opt}) which minimize the first order approximation of the MSE in (4.6):

$$\widetilde{V}_{n_1,n_2} = \frac{(\sigma^2+\tau)^2}{4(\beta-\tau)^4}\frac{\beta^2}{n_1} + \frac{(2\beta\tau-\tau^2+\sigma^2)^2}{4(\beta-\tau)^4}\frac{\sigma^2}{n_2} + \frac{1}{4(\beta-\tau)^2}\left(\frac{\kappa^4-\sigma^4}{n_2}\right) + \frac{2\beta\tau-\tau^2+\sigma^2}{2(\beta-\tau)^3}\frac{\kappa^3}{n_2}, \tag{4.7}$$

subject to the constraint that the total sampling cost $n_1 + n_2 \leq b$, where n_i is the sample size for the i-th population and b is a pre specified total sampling budget. The values (n_1^{opt}, n_2^{opt}), referred to as the first-order allocation which minimize (4.7) are

$$\begin{cases} n_1^{opt} = & \frac{b\beta(\tau^2+\sigma^2)(\Delta-\beta(\tau^2+\sigma^2))}{\Phi} \\ n_2^{opt} = & \frac{b\Delta(\Delta\text{-}\beta(\tau^2+\sigma^2))}{\Phi} \end{cases} \tag{4.8}$$

where

$$\begin{aligned} \Delta^2 &= \beta^2(4\kappa^3\tau + \kappa^4 + 4\tau^2\sigma^2 - \sigma^4) - 2\beta(\kappa^3(3\tau^2 - \sigma^2) + \tau(\kappa^4 + \sigma^2(2\tau^2 - 3\sigma^2))) \\ &\quad + 2\kappa^3\tau(\tau^2 - \sigma^2) + \kappa^4\tau^2 + \tau^4\sigma^2 - 3\tau^2\sigma^4 + \sigma^6 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \Phi = \beta^2(4\kappa^3\tau + \kappa^4 - \tau^4 + 2\sigma^2(\tau^2 - \sigma^2)) - 2\beta(\kappa^3(3\tau^2 - \sigma^2) \\ + \tau(\kappa^4 + \sigma^2(2\tau^2 - 3\sigma^2))) + 2\kappa^3\tau(\tau^2 - \sigma^2) + \kappa^4\tau^2 + \sigma^2(\tau^4 - 3\tau^2\sigma^2 + \sigma^4). \end{aligned} \tag{4.10}$$

Therefore, one can show that the lower bound for the MSE is given by

$$V_{low} = V_{n_1^{opt}, n_2^{opt}} = V^0(b) + O(\tfrac{1}{b^2}),$$

with

$$V^0(b) = \tilde{V}_{n_1^{opt}, n_2^{opt}}. \tag{4.11}$$

4.3 Two Stage Sampling Procedure

Because the first-order allocation (n_1^{opt}, n_2^{opt}) depends on the unknown parameters β, τ, and σ^2, we will propose a two-stage procedure similar to that in Zheng, Seila, and Sriram (1997b) and establish its optimality properties as the total budget goes to infinity.

The objective of the two-stage procedure is to determine the final sample sizes for the two populations that minimize the first-order approximation of the MSE in (4.6) subject to the fixed overall budget (sample size). The procedure works as follows: In stage 1 we select equal-sized samples from each population. Then we estimate n_1^{opt} and n_2^{opt} in (4.8) by $n_1^\star$ and $n_2^\star$ using sample estimates of β, τ, and σ^2 computed from the initial sample. In stage 2, we allocate the remaining budget based on the stage 1 results. The detailed procedure follows:

Stage One:

(1) Start with initial random samples $X_{11}, X_{12}, \ldots, X_{1m_0}$ and $X_{21}, X_{22}, \ldots, X_{2m_0}$ for a suitable m_0 to be defined below.

(2) Compute sample estimates:

$$\widehat{\beta} = \overline{X}_{1m_0},\ \widehat{\tau} = \overline{X}_{2m_0},\ \widehat{\sigma}^2(m_0) = \frac{\sum_{i=1}^{m_0}\left(X_i - \overline{X}_{2m_0}\right)^2}{m_0 - 1}.$$

(3) Let $n_1^\star = \frac{b\widehat{\beta}(\widehat{\tau}^2+\widehat{\sigma}^2)(\widehat{\Delta}-\beta(\widehat{\tau}^2+\widehat{\sigma}^2))}{\widehat{\Phi}}$, and (4.12)

$$n_2^\star = \frac{b\widehat{\Delta}(\widehat{\Delta}\text{-}\widehat{\beta}(\widehat{\tau}^2+\widehat{\sigma}^2))}{\widehat{\Phi}},$$

where $\widehat{\Delta}$ and $\widehat{\Phi}$ are as in (4.9) and (4.10) respectively with β, τ, and σ^2 replaced by their respective estimators.

Stage Two:

(1) Let

$$N_1^\star = \begin{cases} m_0 & \text{if } n_1^\star \le m_0 \\ b - m_0 & \text{if } n_1^\star \ge m_0 \\ n_1^\star & \text{if } m_0 < n_1^\star < b - m_0 \end{cases} \tag{4.13}$$

$$N_2^\star = b - N_1^\star \tag{4.14}$$

and

$$N_1 = [N_1^\star],\, N_2 = b - N_2, \tag{4.15}$$

where $[x]$ is the largest integer less than or equal to x.

(2) Sample $(N_i - m_0)$ more observations from population i, $i = 1, 2$.

(3) Finally, estimate W_q by $\widehat{W}_q$.

4.4 Optimal Properties of the Two Stage Estimator

Theorem 4.3 below shows that the two stage procedure defined in (4.12) to (4.15) is asymptotically risk efficient. Before we state the theorem, we give a list of sufficient conditions needed to prove the theorem. The proof is given in Chapter 6.

Let

$\boldsymbol{Z}^+$ denote the positive integers,

$$T = \frac{\widehat{\tau}}{\widehat{\beta}}, \tag{4.16}$$

$$e = \min(\frac{\tau}{\beta}, \rho_0 - \frac{\tau}{\beta}), \tag{4.17}$$

and

$$D(\epsilon) = \left[\frac{\tau}{\beta} - \epsilon, \frac{\tau}{\beta} + \epsilon\right]. \tag{4.18}$$

The sufficient conditions are:

(1) $EX_2^j < \infty, i = 1, 2,\ j \in \boldsymbol{Z}^+;$ (4.19)

(2) There exists an $\epsilon \in (0, e)$

$$t\{\int_{[T(t)\in D_i(\epsilon)]} T^k(t)dP - (\frac{\tau}{\beta})^k\} = O(1), \text{ as } t \to \infty; \text{ for } k = 1, 2, 3, 4. \qquad (4.20)$$

Theorem 4.3 (Risk Efficiency): *Suppose populations* 1 *and* 2 *satisfy conditions* (4.19) to (4.20). *Let* $.5 < \alpha < 1 - \frac{ln(2)}{ln(b)}$ *and assume that the initial sample size* $m_0 = b^\alpha$. *For* N_i *defined in* (4.15) *and* $h_0 = min\{2 - \alpha, 1 + \alpha\}$ *we have*

$$E(\widehat{W}_q - W_q)^2 = V^0(b) + O(\frac{1}{b^{h_0}}), \text{ as } b \to \infty, \qquad (4.21)$$

where $V^0(b)$ *is defined as in* (4.11).

The two stage procedure outlined in (4.12) to (4.15) is valid for any service time distribution. One interesting special case of the M/G/1 queue is the M/D/1 queue where the service times are deterministic. Our two stage procedure can handle this special case. The values of σ^2, κ^3, and κ^4 are all zero when the service times are deterministic. The result of minimizing the first order approximation of the MSE is that $n_2^{opt} = 0$ indicating the entire budget should be spent sampling the interarrival times. In practice, using the two stage procedure, the experimenter would have already spent m_0 of his budget sampling from the service time distribution to get estimates of β, τ, σ^2, κ^3, and κ^4. The remaining budget would then be allocated to the interarrival time distribution based on the result that $n_2^\star = 0$. The practical implications of using the two stage sampling scheme under various service time distributions is explored further in Chapter 5 of this dissertation.

CHAPTER 5

EMPIRICAL RESULTS FOR THE QUEUEING MODEL

We designed and ran simulations to demonstrate the properties presented in Theorem 4.3, and to evaluate the performance of the M/G/1 model under different service time distributions. Of particular interest was the performance of the M/G/1 sampling scheme when the service time distribution is exponential. The effects of changing parameters such as ρ, traffic intensity, cv, coefficient of variation, and α, used to compute the initial sample size, were also investigated. The simulations used the same hardware and software as those presented in Chapter 3. 10,000 replications were performed for each design point.

The performance of the M/G/1 sampling scheme presented in (4.12) to (4.15) was evaluated in light of the performance of the M/M/1 sampling procedure developed by Zheng, Seila and Sriram (1997b). Table 5.1 is a reproduction of the some of the data in Table 1 of Zheng, Seila and Sriram (1997b). It contains values for $(\widehat{V}_{two} - V_{low}/V_{low})$ as computed using their sampling scheme for the M/M/1 queue described in Chapter 2 of this dissertation.

Table 5.1 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$ in the M/M/1 queue

b	$\rho = 0.50$	$\rho = 0.60$	$\rho = 0.70$	$\rho = 0.80$	$\rho = 0.85$
100	3.084	4.143	2.411	-0.247	-0.801
200	0.799	2.193	2.232	0.079	-0.700
400	0.256	0.522	1.312	0.403	-0.547
800	0.158	0.244	0.547	0.670	-0.325
1000	0.061	0.224	0.411	0.641	-0.256
2000	0.053	0.105	0.186	0.456	0.003
4000	0.031	0.046	0.078	0.222	0.121
8000	0.011	0.013	0.048	0.073	0.188

The results in Table 5.2 were achieved using the M/G/1 sampling scheme presented in (4.12) to (4.15) and assuming an exponential service time distribution with values of ρ in $(0.5, 0.7)$ and $\rho_0 = 0.9$. 95% upper and lower confidence bounds are also provided.

Table 5.2 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$ in the M/G/1 queue

b	LCB	$\rho = .50$	UCB	LCB	$\rho = .60$	UCB	LCB	$\rho = .70$	UCB
100	2.049	2.522	2.994	3.534	3.912	4.291	2.100	2.243	2.385
200	0.066	0.808	1.016	1.718	2.018	2.318	2.097	2.276	2.455
400	0.186	0.242	0.298	0.512	0.624	0.726	1.160	1.315	1.470
800	0.135	0.182	0.228	0.174	0.225	0.275	0.472	0.564	0.656
1000	0.051	0.088	0.125	0.135	0.183	0.231	0.287	0.364	0.441
2000	0.054	0.090	0.125	0.059	0.096	0.133	0.124	0.173	0.223
4000	-0.021	0.009	0.038	0.007	0.039	0.070	0.047	0.082	0.117
8000	-0.017	0.012	0.041	-0.031	-0.002	0.027	0.026	0.061	0.095

b	LCB	$\rho = .80$	UCB	LCB	$\rho = .85$	UCB
100	-0.295	-0.275	-0.797	-0.805	-0.801	-0.797
200	0.011	0.041	0.072	-0.706	-0.701	-0.695
400	0.355	0.401	0.447	-0.552	-0.544	-0.535
800	0.572	0.635	0.698	-0.345	-0.331	-0.317
1000	0.544	0.612	0.680	-0.275	-0.260	-0.244
2000	0.406	0.475	0.543	-0.010	0.016	0.042
4000	0.162	0.209	0.255	0.136	0.173	0.209
8000	0.067	0.104	0.142	0.131	0.172	0.213

The results in Table 5.2 demonstrate the convergence of the estimated MSE, $\widehat{V}_{two}$ to the first order term for the MSE, V_{low} given by (4.11). From Table 5.2, one can see that as the total sampling budget, b, becomes large, $\widehat{V}_{two} - V_{low}$ tends to 0. This empirical data supports the theoretical results claimed in Theorem 4.3. One may observe from Table 5.2 that the convergence of $\widehat{V}_{two} - V_{low}$ to 0 is much slower as the traffic intensity increases. Comparing the results from Table 5.2 with that of Table 5.1, one can see the performance of the M/G/1 sampling procedure is remarkably similar to the results using a scheme based on the M/M/1 queue. There appears to be no degradation in performance by using the more general model. Figure 5.1 displays this information graphically for the traffic intensity $\rho = 0.7$.

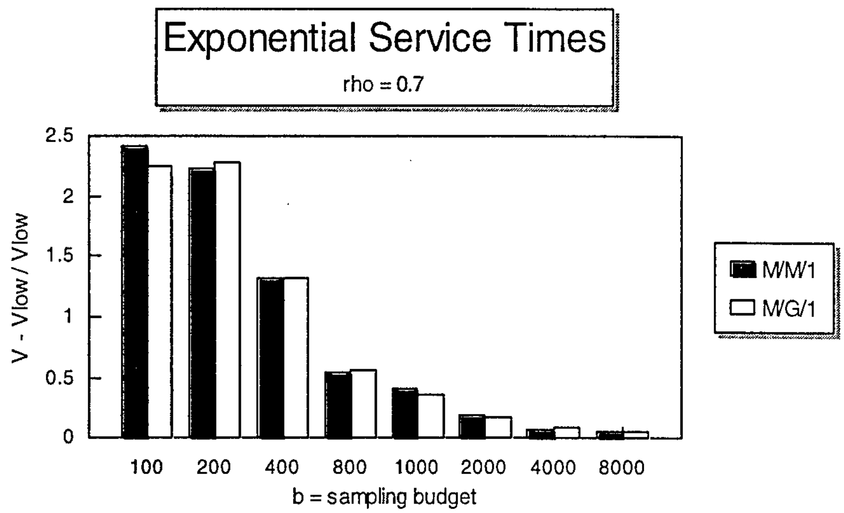

Figure 5.1

When the service times are exponential the coefficient of variation of the service time distribution is 1.0. We conducted simulations using different service time distributions to study the effect of changing the coefficient of variation of the service times. We first looked at cv values greater than one. For these simulations we used a hyperexponential service time distribution. The hyperexponential distribution with three parameters, p, τ_1, and τ_2 can be thought of as being created from two exponential distributions with means τ_1 and τ_2 respectively. To sample from a hyperexponential distribution, one samples from the first exponential distribution with probability p or the second exponential distribution with probability $(1 - p)$.

Table 5.3 provides values of $(\widehat{V}_{two} - V_{low}/V_{low})$ and its 95% confidence intervals for our two stage procedure with various traffic intensities when the coefficient of variation is 2.0. Table 5.4 contains the same information when the coefficient of variation is 5.0.

Table 5.3 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$, CV = 2.0

b	LCB	$\rho = .50$	UCB	LCB	$\rho = .70$	UCB
100	11.304	13.892	16.480	1.676	2.125	2.573
200	8.496	12.201	15.906	3.042	3.513	3.983
400	1.622	2.353	3.085	3.599	4.455	5.311
800	0.521	0.735	0.950	2.304	2.680	3.055
1000	0.350	0.477	0.605	1.791	2.129	2.467
2000	0.191	0.266	0.341	0.689	0.866	1.043
4000	0.113	0.171	0.230	0.268	0.344	0.420
8000	0.011	0.047	0.083	0.110	0.155	0.200

b	LCB	$\rho = .80$	UCB	LCB	$\rho = .90$	UCB
100	-0.343	-0.265	-0.187	-0.931	-0.923	-0.915
200	0.052	0.147	0.243	-0.886	-0.876	-0.866
400	0.679	0.842	1.005	-0.831	-0.816	-0.800
800	0.904	1.084	1.264	-0.794	-0.784	-0.773
1000	0.984	1.124	1.265	-0.777	-0.767	-0.756
2000	0.903	1.028	1.153	-0.737	-0.728	-0.719
4000	0.631	0.736	0.841	-0.692	-0.684	-0.676
8000	0.265	0.335	0.404	-0.656	-0.647	-0.638

Table 5.4 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$, CV = 5.0

b	LCB	$\rho = .50$	UCB	LCB	$\rho = .70$	UCB
100	0.977	1.283	1.589	2.418	2.587	2.757
200	0.327	0.392	0.456	1.801	1.990	2.179
400	0.081	0.123	0.164	0.645	0.754	0.864
800	0.032	0.067	0.101	0.219	0.281	0.343
1000	0.037	0.070	0.103	0.188	0.240	0.292
2000	-0.003	0.026	0.056	0.048	0.085	0.121
4000	0.002	0.031	0.061	0.019	0.051	0.084
8000	-0.019	0.009	0.038	-0.035	-0.006	0.024

b	LCB	$\rho = .80$	UCB	LCB	$\rho = .90$	UCB
100	-0.015	0.009	0.033	-0.943	-0.941	-0.939
200	0.329	0.369	0.409	-0.916	-0.913	-0.911
400	0.578	0.637	0.696	-0.882	-0.879	-0.875
800	0.558	0.630	0.701	-0.839	-0.833	-0.828
1000	0.522	0.597	0.671	-0.821	-0.815	-0.809
2000	0.250	0.305	0.361	-0.781	-0.773	-0.765
4000	0.098	0.140	0.181	-0.732	-0.722	-0.712
8000	0.043	0.077	0.111	-0.693	-0.681	-0.668

We also studied cv values less than one. For these simulations the service times were distributed uniformly. Table 5.5 provides values of $(\widehat{V}_{two} - V_{low}/V_{low})$ and its 95% confidence intervals for our two stage procedure with various traffic intensities when the coefficient of variation is 0.01. Table 5.6 contains the same information when the coefficient of variation is 0.5.

Table 5.5 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$, CV = 0.01

b	LCB	$\rho = .50$	UCB	LCB	$\rho = .70$	UCB
100	0.728	0.831	0.933	2.479	2.703	2.927
200	0.370	0.425	0.479	1.263	1.442	1.622
400	0.225	0.268	0.311	0.450	0.518	0.586
800	0.142	0.177	0.213	0.236	0.282	0.328
1000	0.134	0.169	0.204	0.167	0.210	0.252
2000	0.052	0.084	0.115	0.100	0.135	0.171
4000	0.062	0.094	0.125	0.078	0.111	0.143
8000	0.033	0.063	0.098	0.025	0.055	0.085

b	LCB	$\rho = .80$	UCB	LCB	$\rho = .90$	UCB
100	0.544	0.588	0.613	-0.902	-0.899	-0.896
200	0.861	0.928	0.995	-0.868	-0.864	-0.859
400	0.860	0.945	1.029	-0.819	-0.813	-0.807
800	0.447	0.517	0.587	-0.767	-0.759	-0.751
1000	0.361	0.421	0.481	-0.754	-0.745	-0.736
2000	0.191	0.237	0.282	-0.718	-0.707	-0.696
4000	0.110	0.148	0.186	-0.675	-0.662	-0.650
8000	0.057	0.089	0.120	-0.618	-0.603	-0.587

Table 5.6 Asymptotic Values of $(\widehat{V}_{two} - V_{low}/V_{low})$, CV $= 0.5$

b	LCB	$\rho = .50$	UCB	LCB	$\rho = .70$	UCB
100	0.977	1.283	1.589	2.418	2.587	2.757
200	0.327	0.392	0.456	1.801	1.990	2.179
400	0.081	0.123	0.164	0.645	0.754	0.864
800	0.032	0.067	0.101	0.219	0.281	0.343
1000	0.037	0.070	0.103	0.188	0.240	0.292
2000	-0.003	0.026	0.056	0.048	0.085	0.121
4000	0.002	0.031	0.061	0.019	0.051	0.084
8000	-0.019	0.009	0.038	-0.035	-0.006	0.024

b	LCB	$\rho = .80$	UCB	LCB	$\rho = .90$	UCB
100	-0.015	0.009	0.033	-0.943	-0.941	-0.939
200	0.329	0.369	0.409	-0.916	-0.913	-0.911
400	0.578	0.637	0.696	-0.882	-0.879	-0.875
800	0.558	0.630	0.701	-0.839	-0.833	-0.828
1000	0.522	0.597	0.671	-0.821	-0.815	-0.809
2000	0.250	0.305	0.361	-0.781	-0.773	-0.765
4000	0.098	0.140	0.181	-0.732	-0.722	-0.712
8000	0.043	0.077	0.111	-0.693	-0.681	-0.668

Figures 5.2 and 5.3 display the information in Tables 5.3 through 5.6 graphically. Figure 5.2 shows the effect of traffic intensity on the convergence of the two stage procedure when CV $= 0.5$.

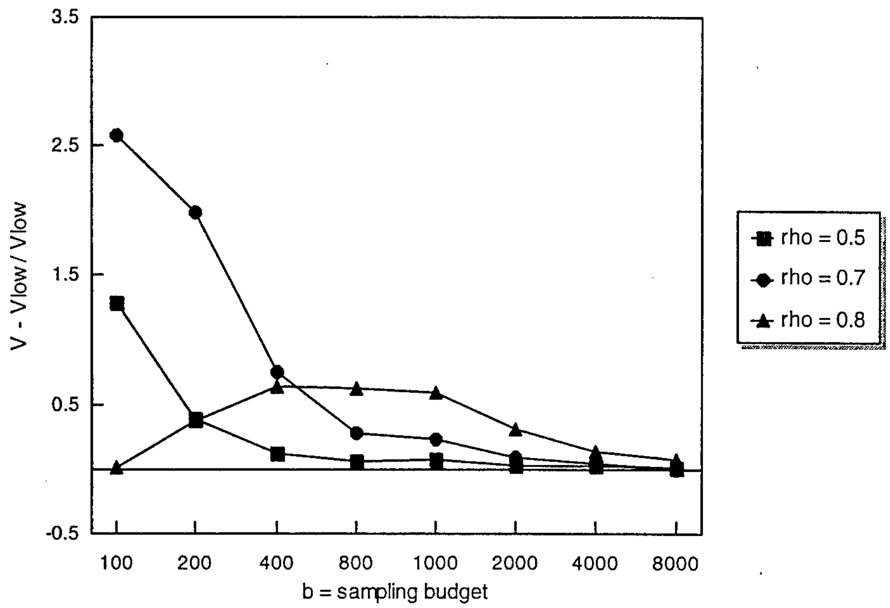

Figure 5.2

Figure 5.2 shows that the MSE of the two stage estimator converges to the theoretical minimum MSE much slower for higher values of traffic intensity.

Figure 5.3 shows the effect of coefficient of variation in the service times on the convergence of the two stage estimator when $\rho = 0.7$.

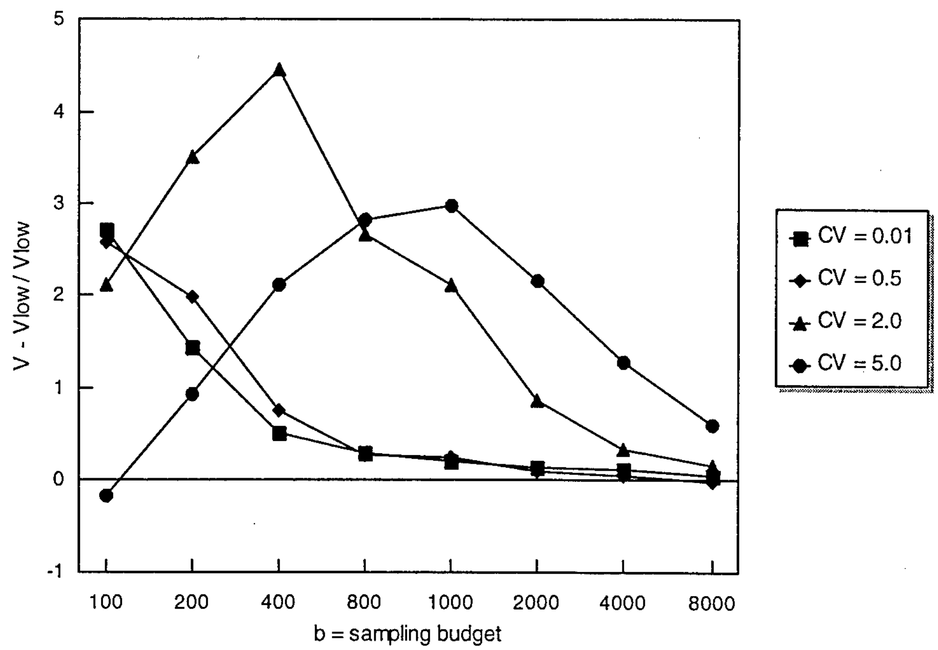

Figure 5.3

Figure 5.3 shows that the MSE of the two stage estimator converges to the theoretical minimum MSE much slower for higher values of coefficient of variation in the service time distribution.

We also conducted simulations to determine the effect of α and therefore initial sample size on the performance of the sampling scheme. For these simulations we chose three different values of α: 0.5, $(0.5 + 1 - \ln(2)/\ln(b))/2$, and $(1 - \ln(2)/\ln(b))$. These values of α are referred to as low, middle, and high respectively. The "high" value of α can actually be thought of as a one stage procedure since $m_0 = b^\alpha = b^{(1-\ln(2)/\ln(b))} = \frac{b}{2}$. The service time distribution for these simulations was exponential. The traffic intensity was 0.5. Table 5.7 provides values for $(\widehat{V}_{two} - V_{low}/V_{low})$ for various values of b. Figure 5.4 displays this information graphically.

Table 5.7 Estimated ($\widehat{V}_{two} - V_{low}/V_{low}$) and Its 95% CI with 10,000 Iterations

M/G/1 Queue with Exponential Services $\rho = 0.5$				
b	α	95% Lbd	$\widehat{V}_{two} - V_{low}/V_{low}$	95% Ubd
100	low	4.042	4.735	5.427
	middle	2.326	2.883	3.440
	high	2.065	2.480	2.896
200	low	1.314	1.793	2.273
	middle	0.577	0.709	0.842
	high	0.591	0.810	1.030
400	low	0.286	0.515	0.743
	middle	0.212	0.287	0.362
	high	0.270	0.330	0.390
800	low	0.093	0.133	0.173
	middle	0.072	0.114	0.155
	high	0.149	0.191	0.234
1000	low	0.116	0.174	0.232
	middle	0.050	0.088	0.125
	high	0.073	0.110	0.148
2000	low	0.040	0.073	0.107
	middle	0.032	0.067	0.102
	high	0.072	0.106	0.140
4000	low	-0.033	-0.004	0.025
	middle	0.005	0.036	0.066
	high	0.044	0.076	0.107

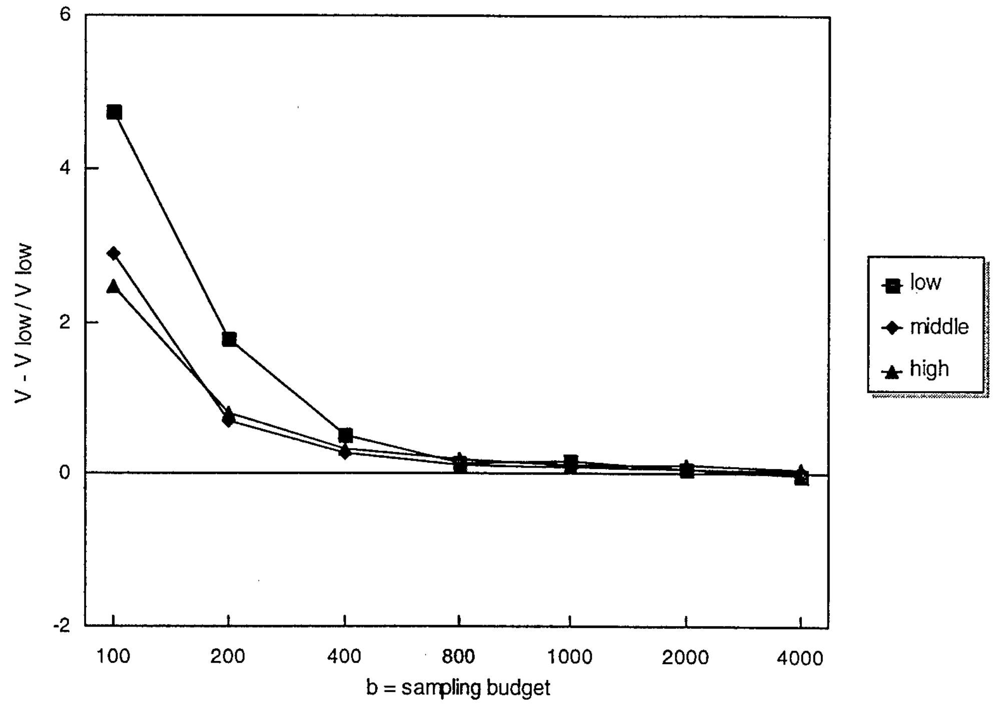

Figure 5.4

Table 5.7 and Figure 5.4 indicate there is no clear preference among the different values for α for all budget sizes, especially beyond $b = 400$. These results do indicate $\alpha = 0.5$ is likely not a good choice for initial sample size at least for small values of b. It is interesting that the two stage estimator does not appear to outperform the one stage estimator represented by the "high" α value. This is because the true minimum of the first order approximation of the MSE in this case is 0.02664911. The MSE of the one stage estimator is 0.0280. This small difference could not be detected because of the sampling error experienced.

All of the above simulations demonstrate that the MSE of the two stage estimator converges to the minimum MSE at reasonable final sample sizes under various service time distributions, traffic intensities, and initial sample sizes.

CHAPTER 6

PROOFS OF THEOREMS

6.1 Proof of Theorem 3.1

Let T_i denote $T_i(m_0)$, $i = 1, 2, 3, 4, 5$ defined in (3.24) through (3.28). Define the following sets for $n_1^\star, n_2^\star$, and $n_3^\star$ defined in (3.8).

$$A_1 = [n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star > m_0],\ A_2 = [n_1^\star > m_0,\ n_2^\star > m_0,\ n_3^\star \leq m_0],$$

$$A_3 = [n_1^\star \leq m_0,\ n_2^\star > m_0,\ n_3^\star > m_0],\ A_4 = [n_1^\star > m_0,\ n_2^\star \leq m_0,\ n_3^\star > m_0],$$

$$A_5 = [n_1^\star > m_0,\ n_2^\star \leq m_0,\ n_3^\star \leq m_0],\ A_6 = [n_1^\star \leq m_0,\ n_2^\star > m_0,\ n_3^\star \leq m_0],$$

$$A_7 = [n_1^\star \leq m_0,\ n_2^\star \leq m_0,\ n_3^\star > m_0].$$

Clearly, A_1, A_2, ..., A_7 are disjoint and the union of these sets is the sample space. Our proof of Theorem 3.1 depends on the following lemmas.

Lemma 6.1.1: *Assume conditions* (3.29) to (3.31). *For β in condition* (3.30), *let $\frac{4}{\beta+4} < \alpha < 1 - \frac{ln(3)}{ln(b)}$ and $m_0 = b^\alpha$. Then, for sets A_1 through A_7 defined above we have, as $b \longrightarrow \infty$,*

$$P(A_l) = O(\frac{1}{b^{\alpha\beta}}),\ l = 2, 3, ..., 7, \tag{6.1}$$

$$E(T_1)I_{A_1} = \frac{\mu_1(\sigma_2 + \sigma_3)}{\sigma_1(\mu_2 + \mu_3)} + O(\frac{1}{b^{\alpha(1+\beta)}}) + O(\frac{1}{b^{(1+\alpha)}}), \tag{6.2}$$

$$E(T_2 + T_3) = \frac{\sigma_3}{\sigma_2} + \frac{\sigma_1(\mu_2 + \mu_3)}{\mu_1\sigma_2} + O(\frac{1}{b^{\alpha(1+\beta)}}) + O(\frac{1}{b^{(1+\alpha)}}), \tag{6.3}$$

$$E(T_4 + T_5) = \frac{\sigma_2}{\sigma_3} + \frac{\sigma_1(\mu_2 + \mu_3)}{\mu_1\sigma_3} + O(\frac{1}{b^{\alpha(1+\beta)}}) + O(\frac{1}{b^{(1+\alpha)}}), \tag{6.4}$$

and

$$E[(T_1)^l I_{A_1}] = O(1), \tag{6.5}$$

$$E[(T_2+T_3)^l I_{A_1}] = O(1), \tag{6.6}$$

$$E[(T_4+T_5)^l I_{A_1}] = O(1), \quad \text{for } l = 2,3,4. \tag{6.7}$$

Proof. For (6.1), we only prove the case $l = 2$ as the rest of the cases can be proved in a similar way. Now, for $D_i(\epsilon)$ defined in (3.14) to (3.18) we have

$$\begin{aligned}
P(A_2) &\le P(n_3^\bigstar \le m_0)\\
&= P\Big(\tfrac{b\bar{X}_1 S_3}{\bar{X}_1(S_2+S_3)+S_1(\bar{X}_2+\bar{X}_3)} \le m_0\Big)\\
&= P\Big(\tfrac{\bar{X}_1(S_2+S_3)+S_1(\bar{X}_2+\bar{X}_3)}{b\bar{X}_1 S_3} \ge \tfrac{1}{b^\alpha}\Big)\\
&= P\Big(\tfrac{1}{b} + \tfrac{\bar{X}_1 S_2}{b\bar{X}_1 S_3} + \tfrac{S_1(\bar{X}_2+\bar{X}_3)}{b\bar{X}_1 S_3} \ge \tfrac{1}{b^\alpha}\Big)\\
&= P\Big(\tfrac{S_2}{S_3} + \tfrac{S_1(\bar{X}_2+\bar{X}_3)}{\bar{X}_1 S_3} \ge \tfrac{b-b^\alpha}{b^\alpha}\Big)\\
&\le P\Big(\tfrac{S_2}{S_3} \ge \tfrac{b-b^\alpha}{2b^\alpha}\Big) + P\Big(\tfrac{S_1(\bar{X}_2+\bar{X}_3)}{\bar{X}_1 S_3} \ge \tfrac{b-b^\alpha}{2b^\alpha}\Big)\\
&= P\big(T_4 \ge \tfrac{b-b^\alpha}{2b^\alpha}\big) + P\big(T_5 \ge \tfrac{b-b^\alpha}{2b^\alpha}\big)\\
&= P\Big(T_4 - \tfrac{\sigma_2}{\sigma_3} \ge \tfrac{b-b^\alpha}{2b^\alpha} - \tfrac{\sigma_2}{\sigma_3}\Big) + P\Big(T_5 - \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} \ge \tfrac{b-b^\alpha}{2b^\alpha} - \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3}\Big)\\
&\le P\Big(\big|T_4 - \tfrac{\sigma_2}{\sigma_3}\big| \ge \tfrac{b-b^\alpha}{2b^\alpha} - \tfrac{\sigma_2}{\sigma_3}\Big) + P\Big(\big|T_5 - \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3}\big| \ge \tfrac{b-b^\alpha}{2b^\alpha} - \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3}\Big)\\
&= P[T_4 \notin D_4(\epsilon)] + P[T_5 \notin D_5(\epsilon)]\\
&= O(\tfrac{1}{b^{\alpha\beta}}),
\end{aligned}$$

where the next to last step follows since for $\alpha < 1$, $\frac{b-b^\alpha}{2b^\alpha} \to \infty$ as $b \to \infty$, and the last step follows from condition (3.30) and since $m_0 = b^\alpha$. Hence (6.1) follows.

As for (6.2) through (6.7), let β and ϵ_0 satisfy conditions (3.30) and (3.31) respectively, with
$0 < \epsilon_0 < \min\Big\{\frac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)}, \frac{\sigma_3}{\sigma_2} + \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2}, \frac{\sigma_2}{\sigma_3} + \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3}\Big\}$ and

$$B = \{T_1 \in D_1(\epsilon_0), T_2 \in D_2(\epsilon_0), T_3 \in D_3(\epsilon_0), T_4 \in D_4(\epsilon_0), T_5 \in D_5(\epsilon_0)\}.$$

Since

$$\begin{aligned}
A_1 &= \{n_1^\bigstar > b^\alpha, n_2^\bigstar > b^\alpha, n_3^\bigstar > b^\alpha\}\\
&= \{b - 2b^\alpha > n_1^\bigstar > b^\alpha, b - 2b^\alpha > n_2^\bigstar > b^\alpha, b - 2b^\alpha > n_3^\bigstar > b^\alpha\}\\
&= \big\{\tfrac{2b^\alpha}{b-2b^\alpha} < T_1 < \tfrac{b-b^\alpha}{b^\alpha}, \tfrac{2b^\alpha}{b-2b^\alpha} < T_2+T_3 < \tfrac{b-b^\alpha}{b^\alpha}, \tfrac{2b^\alpha}{b-2b^\alpha} < T_4+T_5 < \tfrac{b-b^\alpha}{b^\alpha}\big\},
\end{aligned}$$

and for $\alpha < 1$, $\frac{2b^\alpha}{b-2b^\alpha} \to 0$ and $\frac{b-b^\alpha}{b^\alpha} \to \infty$, as $b \to \infty$, it follows for large b, that

$$
\begin{aligned}
B \subseteq & \left\{ \tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} - \epsilon_0 < T_1 < \tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} + \epsilon_0 \right\} \\
& \cap \left\{ \tfrac{\sigma_3}{\sigma_2} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} - \epsilon_0 < T_2 + T_3 < \tfrac{\sigma_3}{\sigma_2} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} + \epsilon_0 \right\} \\
& \cap \left\{ \tfrac{\sigma_2}{\sigma_3} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} - \epsilon_0 < T_4 + T_5 < \tfrac{\sigma_2}{\sigma_3} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} + \epsilon_0 \right\} \\
& \subseteq A_{1.}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &\leq E[(T_1)^l I_{A_1}] - E[(T_1)^l I_B] \\
&\leq (\tfrac{b}{b^\alpha})^l P(A_1 - B), \\
&= O(\tfrac{1}{b^{(\alpha\beta+\alpha l - l)}}),\ l = 1, 2, 3, 4,
\end{aligned}
\tag{6.8}
$$

as b $\longrightarrow \infty$, where the last step follows from condition (3.30) and since $m_0 = b^\alpha$. Note that this expression is also true for $E[(T_2+T_3)^l I_{A_1}] - E[(T_2+T_3)^l I_B]$ and for $E[(T_4+T_5)^l I_{A_1}] - E[(T_4+T_5)^l I_B]$.

Next we show that conditions (3.30) and (3.31) imply that for each $i = 1, 2, 3, 4, 5$

$$
\int_{[T_i(t)\in D_i(\epsilon)]} T_i^l(t) dP = R_i^l + O(\frac{1}{t}),\ \text{as } t \longrightarrow \infty, \tag{6.9}
$$

for $l = -3, -2, -1, 1, 2, 3$. To this end, we only show (6.9) for the case $l = 3$ as similar arguments yield the result for the cases $l = -3, -2, -1, 1, 2$. Note that for each $i = 1, 2, 3, 4, 5$

$$
R_i^3\{1 - (3/4)R_i^4[T_i^4 - R_i^4]\} \leq T_i^3(t) \leq R_i^3\{1 + (3/4)R_i^4[T_i^4 - R_i^4]\},
$$

if $T_i(t) \in D_i(\epsilon)$. Hence, by conditions (3.30) and (3.31)

$$
\int_{[T_i(t)\in D_i(\epsilon)]} T_i^l(t) dP = R_i^l + O(\tfrac{1}{t}),\ \text{as } t \longrightarrow \infty.
$$

Now, from (6.9) and (3.30) we have

$$
\begin{aligned}
E[(T_1)I_B] &= E[T_1 I_{[T_1(t)\in D_1(\epsilon)]}]\prod_{i\neq 1} P(T_i \in D_i(\epsilon) \\
&= \left(\tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} + O(\tfrac{1}{b^\alpha}) \right) \left(1 - O(\tfrac{1}{b^{\alpha\beta}}\right)^4 \\
&= \tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)} + O(\tfrac{1}{b^\alpha})
\end{aligned}
\tag{6.10}
$$

Similarly,

$$\begin{aligned} E[(T_2+T_3)I_B] &= \big(E[T_2 I_{[T_2(t)\in D_2(\epsilon)]}] + E[T_3 I_{[T_3(t)\in D_3(\epsilon)]}]\big)\prod_{i\neq 2,3} P(T_i \in D_i(\epsilon) \\ &= \Big(\tfrac{\sigma_3}{\sigma_2} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} + O(\tfrac{1}{b^\alpha})\Big)\big(1 - O(\tfrac{1}{b^{\alpha\beta}}\big)^3 \\ &= \tfrac{\sigma_3}{\sigma_2} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2} + O(\tfrac{1}{b^\alpha}) \end{aligned}$$

and

$$\begin{aligned} E[(T_4+T_5)I_B] &= \big(E[T_4 I_{[T_4(t)\in D_4(\epsilon)]}] + E[T_5 I_{[T_5(t)\in D_5(\epsilon)]}]\big)\prod_{i\neq 4,5} P(T_i \in D_i(\epsilon) \\ &= \Big(\tfrac{\sigma_2}{\sigma_3} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} + O(\tfrac{1}{b^\alpha})\Big)\big(1 - O(\tfrac{1}{b^{\alpha\beta}}\big)^3 \\ &= \tfrac{\sigma_2}{\sigma_3} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_3} + O(\tfrac{1}{b^\alpha}) \end{aligned}$$

Now, (6.2) follows from (6.8) and (6.10). Assertions (6.3) and (6.4) follow similarly. The results in (6.5) to (6.7) can be obtained using (6.8), (6.9) and arguments as in (6.10). □

Lemma 6.1.2: *Suppose the assumptions of Lemma* 6.1.1 *hold. Let* $h_1 = min(\alpha(1+\beta/2), 2-\alpha)$ *for* α *and* β *in Lemma* 6.1.1. *Then for* N_i, $i = 1,2,3$ *defined in* (3.12) *we have, as* $b \to \infty$,

$$E(\overline{X}_{1N_1} - \mu_1)^2 = \sigma_1^2 E\frac{1}{N_1^\star} + O(\frac{1}{b^{h_1}}) \tag{6.11}$$

$$E(\overline{X}_{2N_2} + \overline{X}_{3N_3} - (\mu_2+\mu_3))^2 = \sigma_2^2 E\frac{1}{N_2^\star} + \sigma_3^2 E\frac{1}{N_3^\star} + O(\frac{1}{b^{h_1}}) \tag{6.12}$$

and

$$E(\overline{X}_{1N_1} - \mu_1)^p(\overline{X}_{2N_2} + \overline{X}_{3N_3} - (\mu_2+\mu_3))^q = O(\frac{1}{b^{h_1}}), \tag{6.13}$$

for $p,\ q = 1,2$.

Proof.
To prove (6.11), note that since the three populations are independent and $N_1(\geq m_0)$ is an integer-valued r.v. measurable w.r.t. $(X_{11}, X_{12}, ..., X_{1m_0}, X_{21}, X_{22}, ..., X_{2m_0}, X_{31}, X_{32}, ..., X_{3m_0})$, by direct calculations it follows that

$$E(\overline{X}_{1N_1}-\mu_1)^2=\sigma_1^2E(\tfrac{1}{N_1})+E\frac{[\sum\limits_{j=1}^{m_0}(X_{1j}-\mu_1)]^2}{N_1^2}-m_0\sigma_1^2E(\tfrac{1}{N_1^2})$$
$$=\sigma_1^2E(\tfrac{1}{N_1^\star})+R_b, \tag{6.14}$$

where

$$R_b=\sigma_1^2E[(N_1^\star-N_1)/(N_1N_1^\star)]+E[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2N_1^{-2}-m_0\sigma_1^2E(\tfrac{1}{N_1^2}). \tag{6.15}$$

We will show that for h_1 defined in the Lemma

$$E\big|R_b\big|=O(\frac{1}{b^{h_1}})\ \text{ as } b\to\infty. \tag{6.16}$$

Since $N_1=[N_1^\star]$, $0\le(N_1^\star-N_1)/(N_1N_1^\star)\le 2(N_1^\star)^{-2}$. Moreover, it suffices to show that the last two terms on the right hand side of (6.15) are $O(\frac{1}{b^{h_1}})$ with N_1 replaced by $N_1^\star$. Now, for $N_1^\star$in (3.9), $n_1^\star$ in (3.8) and sets $A_1,\ldots,A_7$ defined earlier

$$E(1/N_1^\star)^2=b^{-2}E[(1+T_1)^2I_{A_1}]+E[(1/N_1^\star)^2I_{A_1^c}], \tag{6.17}$$

where, since $N_1^\star\ge m_0=b^\alpha$,

$$E[(1/N_1^\star)^2I_{A_1^c}]\le m_0^{-2}P(A_2\cup\ldots\cup A_7)$$
$$=O(\tfrac{1}{b^{\alpha(\beta+2)}})\ \text{ as } b\to\infty, \tag{6.18}$$

by (6.1). Moreover, by (6.2) and (6.5)

$$b^{-2}E[(1+T_1)^2I_{A_1}]=O(b^{-2})\ \text{ as } b\to\infty. \tag{6.19}$$

Hence, from (6.17) to (6.19) and since $\alpha>\frac{4}{\beta+4}$

$$E(1/N_1^\star)^2=O(\frac{1}{b^{\alpha(\beta+2)}})\ \text{ as } b\to\infty. \tag{6.20}$$

From this and since $m_0=b^\alpha$, the third term on the right hand side of (6.15)

$$m_0\sigma_1^2E(\frac{1}{N_1^2})=O(\frac{1}{b^{\alpha(\beta+1)}})\ \text{ as } b\to\infty. \tag{6.21}$$

Finally, as in (6.17)

$$E[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2N_1^{-2} = b^{-2}E[[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2(1+T_1)^2I_{A_1}] + E[\{[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2N_1^{-2}\}I_{A_1^c}]. \tag{6.22}$$

Since $N_1^\star \geq m_0 = b^\alpha$ and $E[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^4 = O(m_0^2)$, by the Cauchy-Schwarz inequality and (6.1)

$$E[\{[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2N_1^{-2}\}I_{A_1^c}] = O(\frac{1}{b^{\alpha(1+\beta/2)}}) \tag{6.23}$$

as $b \to \infty$. A similar argument using (6.5) yields

$$b^{-2}E[[\sum_{j=1}^{m_0}(X_{1j}-\mu_1)]^2(1+T_1)^2I_{A_1}] = O(\frac{1}{b^{(2-\alpha)}}) \tag{6.24}$$

as $b \to \infty$. The assertion (6.16) now follows from (6.17) to (6.24). Hence the assertion (6.11).

For (6.12) note that

$$\begin{aligned}&E(\overline{X}_{2N_2}+\overline{X}_{3N_3}-(\mu_2+\mu_3))^2\\&=E(\overline{X}_{2N_2}-\mu_2)^2+E(\overline{X}_{3N_3}-\mu_3)^2\\&-2(\overline{X}_{2N_2}-\mu_2)(\overline{X}_{3N_3}-\mu_3)\\&=\sigma_2^2E\tfrac{1}{N_2^\star}+\sigma_3^2E\tfrac{1}{N_3^\star}+O(\tfrac{1}{b^{h_1}})\end{aligned} \tag{6.25}$$

where the last statement follows from the same arguments used to prove assertion (6.11).

For (6.13) with $p=1,\ q=1$, first note that

$$\begin{aligned}&E(\overline{X}_{1N_1}-\mu_1)(\overline{X}_{2N_2}+\overline{X}_{3N_3}-(\mu_2+\mu_3))\\&=E(\overline{X}_{1N_1}-\mu_1)(\overline{X}_{2N_2}-\mu_2)+E(\overline{X}_{1N_1}-\mu_1)(\overline{X}_{3N_3}-\mu_3)\end{aligned} \tag{6.26}$$

so it suffices to show

$$E(\overline{X}_{1N_1} - \mu_1)(\overline{X}_{2N_2} - \mu_2) = O(\frac{1}{b^{h_1}}) \tag{6.27}$$

As in (6.14), we have

$$E(\overline{X}_{1N_1} - \mu_1)(\overline{X}_{2N_2} - \mu_2) = E[\sum_{j=1}^{m_0}(X_{1j} - \mu_1)]E[\sum_{j=1}^{m_0}(X_{2j} - \mu_2)]/(N_1N_2). \tag{6.28}$$

Once again, it suffices to show that the right hand side of (6.28) is $O(\frac{1}{b^{h_1}})$ with N_1 and N_2 replaced by $N_1^\star$ and $N_2^\star$, respectively. As in (6.22), the right hand side of (6.28) is

$$\begin{aligned}
&= b^{-2}E[\sum_{j=1}^{m_0}(X_{1j} - \mu_1)]E[[\sum_{j=1}^{m_0}(X_{2j} - \mu_2)](1+T_1)(1+T_2+T_3)I_{A_1}] \\
&+ E\{[\sum_{j=1}^{m_0}(X_{1j} - \mu_1)]E[[\sum_{j=1}^{m_0}(X_{2j} - \mu_2)]/(N_1^\star N_2^\star)\}I_{A_1^c}] \\
&= O(\tfrac{1}{b^{(2-\alpha)}}) + O(\tfrac{1}{b^{\alpha(1+\beta/2)}}) \text{ as } b \to \infty,
\end{aligned} \tag{6.29}$$

using arguments similar to (6.23) and (6.24). Hence the assertion (6.27), and hence the required result in (6.13) with $p = q = 1$. □

Proof of Theorem 3.1.
Write :

$$\begin{aligned}
\overline{X}_{1N_1}(\overline{X}_{2N_2} + \overline{X}_{3N_3}) - \mu_1(\mu_2+\mu_3) = {} & (\overline{X}_{1N_1} - \mu_1)(\mu_2+\mu_3) \\
& + (\overline{X}_{2N_2} + \overline{X}_{3N_3} - (\mu_2+\mu_3))\mu_1 \\
& + (\overline{X}_{1N_1} - \mu_1)(\overline{X}_{2N_2} + \overline{X}_{3N_3} - (\mu_2+\mu_3))
\end{aligned} \tag{6.30}$$

Then by Lemma 6.1.2, we have

$$\begin{aligned}
&E(\overline{X}_{1N_1}(\overline{X}_{2N_2} + \overline{X}_{3N_3}) - \mu_1(\mu_2+\mu_3))^2 = \\
&(\mu_2+\mu_3)^2\sigma_1^2 E\tfrac{1}{N_1^\star} + \mu_1^2\sigma_2^2 E\tfrac{1}{N_2^\star} + \mu_1^2\sigma_3^2 E\tfrac{1}{N_3^\star} + O(\tfrac{1}{b^{h_1}}),
\end{aligned} \tag{6.31}$$

as $b \to \infty$. By the definition of $N_1^\star$in (3.9) and arguments as in (6.17) we have by Lemma 6.1.1 that

$$\begin{aligned}
E\tfrac{1}{N_1^\star} &= \tfrac{1}{b}E[(1+T_1)I_{A_1}] + E[(\tfrac{1}{N_1^\star})I_{A_1^c}] \\
&= \tfrac{1}{b}\left(1 + \tfrac{\mu_1(\sigma_2+\sigma_3)}{\sigma_1(\mu_2+\mu_3)}\right) + O(\tfrac{1}{b^{\alpha(1+\beta)}}) + O(\tfrac{1}{b^{(1+\alpha)}}).
\end{aligned} \tag{6.32}$$

Similarly,

$$E\tfrac{1}{N_2^\star} = \tfrac{1}{b}\left(1 + \tfrac{\sigma_3}{\sigma_2} + \tfrac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma_2}\right) + O(\tfrac{1}{b^{\alpha(1+\beta)}}) + O(\tfrac{1}{b^{(1+\alpha)}}) \tag{6.33}$$

and

$$E\frac{1}{N_3^\star} = \frac{1}{b}\left(1 + \frac{\sigma_2}{\sigma_3} + \frac{\sigma_1(\mu_2+\mu_3)}{\mu_1\sigma 3}\right) + O(\frac{1}{b^{\alpha(1+\beta)}}) + O(\frac{1}{b^{(1+\alpha)}}). \quad (6.34)$$

Substituting these expressions into (6.30), we have (3.32). Hence the theorem. □

6.2 Proofs of Chapter 4 Theorems

Proof of Theorem 4.1.
For (4.2), note that

$$E\left|\widehat{W}_{1q}\right| = E_{Y_1,Y_2,\ldots,Y_{n_2}}[E_{\overline{X}}[\left|\widehat{W}_{1q}\right|\Big|Y_1, Y_2, \ldots, Y_{n_2}],$$

and that

$$E_{\overline{X}}[\left|\widehat{W}_{1q}\right|\Big|Y_1, Y_2, \ldots, Y_{n_2}] = \frac{n_1}{\Gamma(n_1)\beta^{n_1}}\int_0^\infty \frac{s_y^2 + \overline{y}^2}{2|x - \overline{y}|}\,(n_1 x)^{n_1-1}e^{-n_1 x/\beta}\,dx = +\infty.$$

Therefore,

$$E\left|\widehat{W}_{1q}\right| = +\infty.$$

(4.3) now follows since

$E\left|\widehat{W}_{1q}\right| = +\infty \Rightarrow E\widehat{W}_{1q}^2 = +\infty.$ □

Proof of Theorem 4.2.

For (4.6), let $D_1(\epsilon_0) = [\beta - \epsilon_0, \beta + \epsilon_0]$, $D_2(\epsilon_0) = [\tau - \epsilon_0, \tau + \epsilon_0]$, $B = [\overline{X}_{1n_1} \in D_1(\epsilon_0), \overline{X}_{2n_2} \in D_2(\epsilon_0)]$, and $A = \left[\frac{\overline{X}_{2n_2}}{\overline{X}_{1n_1}} \leq \rho_0\right]$. Note that there exists an $\epsilon_0 \in (0, \frac{\beta(\rho_0-\rho)}{1-\rho_0})$, such that $B \subseteq A$. Observe that for any $\omega > 1$, and any $\epsilon \in (0, \frac{\beta(\rho_0-\rho)}{1-\rho_0})$,

$$P\left[\overline{X}_{1n_1} \notin D_1(\epsilon)\right] = O(\frac{1}{n_1^\omega}), \text{ and } P\left[\overline{X}_{2n_2} \notin D_2(\epsilon)\right] = O(\frac{1}{n_2^\omega}). \quad (6.35)$$

We have:

$$E(\widehat{W}_q - W_q)^2 = E(\widehat{W}_q - W_q)^2 I_A + E(\widehat{W}_q - W_q)^2 I_{A^c} \quad (6.36)$$

First, note the second term in (6.36)

$$E(\widehat{W}_q - W_q)^2 I_{A^c} = O(\frac{1}{n_1^{\omega/2}}) + O(\frac{1}{n_2^{\omega/2}})$$

by the Cauchy-Schwarz inequality.
Now note that the first term in (6.36) is

$$\begin{aligned}
&E(\widehat{W}_q - W_q)^2 I_A - E(\widehat{W}_q - W_q)^2 I_B + E(\widehat{W}_q - W_q)^2 I_B \\
&\quad = E(\widehat{W}_q - W_q)^2 I_{A-B} + E(\widehat{W}_q - W_q)^2 I_B \\
&\quad \leq (E(\widehat{W}_q - W_q)^4)^{1/2}(E(I_{A-B}))^{1/2} + E(\widehat{W}_q - W_q)^2 I_B
\end{aligned}$$

$$
\begin{aligned}
&= (E(\widehat{W}_q - W_q)^4)^{1/2}(P(A-B))^{1/2} + E(\widehat{W}_q - W_q)^2 I_B \\
&= O(1)\big(P\big[\overline{X}_{1n_1} \notin D_1(\epsilon) \cup \overline{X}_{2n_2} \notin D_2(\epsilon)\big]\big)^{1/2} + E(\widehat{W}_q - W_q)^2 I_B \\
&\leq O(\tfrac{1}{n_1^{\omega/2}}) + O(\tfrac{1}{n_2^{\omega/2}}) + E(\widehat{W}_q - W_q)^2 I_B, \qquad (6.37)
\end{aligned}
$$

for any $\omega > 1$.

Thus by expanding for the function $\widehat{W}_q$ in a Taylor series on B, and (6.37), we have (4.6).
For (4.5), note that similar to (6.37), we have, for any $\omega > 1$,

$$
\begin{aligned}
E(\widehat{W}_q - W_q)I_A &= E(\widehat{W}_q - W_q)I_B + O(\tfrac{1}{n_1^{\omega/2}}) + O(\tfrac{1}{n_2^{\omega/2}}) \\
&= O(\tfrac{1}{n_1}) + O(\tfrac{1}{n_2}). \qquad (6.38)
\end{aligned}
$$

(4.5) follows from (6.35) and (6.38). □

Our proof of Theorem 4.3 depends on the following three lemmas. Let $T(n)$, e, and $D(\epsilon)$ be defined as in (4.16) to (4.18).

Lemma 6.2.1: *If $X_{11}, X_{12}, \ldots X_{1n}$ are i.i.d. exponential random variables with mean β and $X_{21}, X_{22}, \ldots X_{2n}$ are i.i.d. random variables with mean τ and $finite$ variance σ^2 such that* (4.19) *holds, then*

For any $\omega > 1$ and any fixed $\epsilon \in (0, e)$,

$$n^{\omega} P[T(n) \notin D(\epsilon)] = O(1) \qquad (6.39)$$

$$n^{\omega} P[s^2(n) \notin D_3(\epsilon)] = O(1) \qquad (6.40)$$

For N_i defined as in (4.15) with $m_0 = b^{\alpha}$, we have:

$$E\frac{1}{N_1} = \frac{\Phi}{b\beta(\tau^2+\sigma^2)(\Delta - \beta(\tau^2+\sigma^2))} + O\frac{1}{b^{(1+\alpha)}} \qquad (6.41)$$

$$E\frac{1}{N_2} = \frac{\Phi}{b\Delta(\Delta\text{-}\beta(\tau^2+\sigma^2))} + O\frac{1}{b^{(1+\alpha)}} \qquad (6.42)$$

where Δ is given in (4.9) and Φ is given in (4.10).

Proof. (6.39) and (6.40) are proved in Zheng, Seila and Sriram (1995). As for (6.41) it suffices to show this result with N_1 replaced by $N_1^{\star}$. Then using arguments as in (6.17) we have:

$$E\frac{1}{N_1^\star} = E\frac{\widehat{\Phi}}{b\widehat{\beta}(\widehat{\tau}^2+\widehat{\sigma}^2)(\widehat{\Delta}-\widehat{\beta}(\widehat{\tau}^2+\widehat{\sigma}^2))} I_{[m_0<n_1^\star<b-m_0]} + O\frac{1}{b^{\alpha(1+\beta)}}$$

$$= \frac{\Phi}{b\beta(\tau^2+\sigma^2)(\Delta-\beta(\tau^2+\sigma^2))} + O(\frac{1}{b^{h_1}}), \tag{6.43}$$

by (4.20) where $h_1 = \min(\alpha(1+\beta), 1+\alpha)$. Hence we have (6.41) noting that ω is arbitrary. (6.42) is proved similarly. □

Lemma 6.2.2: *If N_1 and N_2 are defined as in* (4.15), $\alpha \in (0.5, 1 - \frac{\ln(2)}{\ln(b)})$, (n_1^{opt}, n_2^{opt}) *are defined as in* (4.8) *and $h_0 = min(1+\alpha, 2-\alpha)$, then:*

$$E(\overline{X}_{1N_1} - \beta)^2 = \beta^2 \frac{1}{n_1^{opt}} + O(\frac{1}{b^{h_0}}) \tag{6.44}$$

$$E(\overline{X}_{2N_2} - \tau)^2 = \sigma^2 \frac{1}{n_2^{opt}} + O(\frac{1}{b^{h_0}}) \tag{6.45}$$

$$E(s^2(N_2) - \sigma^2)^2 = (\kappa^4 - \sigma^4)\frac{1}{n_2^{opt}} + O(\frac{1}{b^{h_0}}) \tag{6.46}$$

$$E(\overline{X}_{1N_1} - \beta)^p(\overline{X}_{2N_2} - \tau)^q(s^2(N_2) - \sigma^2)^r = O(\frac{1}{b^{h_0}}) \tag{6.47}$$

for $p, q, r = 0, 1, 2$, such that $(p,q,r) \neq (2,0,0)$, $(0,2,0)$, $(0,0,2)$, and $(0,1,1)$.

$$E(\overline{X}_{2N_2} - \tau)(s^2(N_2) - \sigma^2) = \kappa^3 \frac{1}{n_2^{opt}} + O(\frac{1}{b^{h_0}}) \tag{6.48}$$

Proof. For (6.44), using similar arguments to prove Lemma 6.1.2 we have

$$E(\overline{X}_{1N_1} - \beta)^2 = \beta^2 E(\frac{1}{N_1^\star}) + O(\frac{1}{b^{h_1}}) \tag{6.49}$$

where $h_1 = \min(\alpha(1+\omega/2), 2-\alpha)$. Now using (6.43) we have

$$E(\frac{1}{N_1^\star}) = E(\frac{1}{n_1^{opt}}) + O(\frac{1}{b^{(1+\alpha)}}) \tag{6.50}$$

Hence (6.44). (6.45), (6.46) and (6.48) are shown by similar arguments. (6.47) is shown using similar arguments used to prove (6.13). □

Lemma 6.2.3: *For any $\omega > 1$ and $0 < \epsilon < 1$, the following hold*

$$P\left[\overline{X}_{1N_1} \notin D_1(\epsilon)\right] = O(\frac{1}{b^{\alpha\omega}}) \tag{6.51}$$

$$P\left[\overline{X}_{2N_2} \notin D_2(\epsilon)\right] = O(\frac{1}{b^{\alpha\omega}}) \tag{6.52}$$

$$P\left[\overline{X}_{2N_2} > \rho_0 \overline{X}_{1N_1}\right] = O(\frac{1}{b^{\alpha\omega}}) \tag{6.53}$$

$$P\left[s^2(N_2) \notin D_3(\epsilon)\right] = O(\frac{1}{b^{\alpha\omega}}) \tag{6.54}$$

Proof. (6.51) to (6.53) are proved in Lemma 3 of Zheng, Seila, and Sriram (1997b). (6.54) is proved in a similar manner. □

Proof of Theorem 4.3.

Let $A = [\overline{X}_{2N_2} \leq \rho_0 \overline{X}_{1N_1}]$. By the definition of $\widehat{W}_q$, we have

$$E\left(\widehat{W}_q - Wq\right)^2 = E\left[\frac{S^2 + \overline{X}_{2N_2}^2}{2\left(\overline{X}_{1N_1} - \overline{X}_{2N_2}\right)} - \frac{\sigma^2+\tau^2}{2(\beta-\tau)}\right]^2 I_A$$

$$+ E\left[\frac{\rho_0 \overline{X}_{2N_2}\left(\frac{S^2}{\overline{X}_{2N_2}^2}+1\right)}{2(1-\rho_0)} - W_q\right]^2 I_{A^c}. \tag{6.55}$$

We first consider the second term in (6.55).

$$E\left[\frac{\rho_0 \overline{X}_{2N_2}\left(\frac{S^2}{\overline{X}_{2N_2}^2}+1\right)}{2(1-\rho_0)} - W_q\right]^2 I_{A^c}$$

$$= \left(\frac{\rho_0}{2(1-\rho_0)}\right)^2 E\left[\left(\frac{s^2}{\overline{X}_{2N_2}} + \overline{X}_{2N_2}\right) - \frac{2(1-\rho_0)}{\rho_0} W_q\right]^2 I_{A^c}$$

$$= \left(\frac{\rho_0}{2(1-\rho_0)}\right)^2 \left\{E\left[\left(\frac{s^2}{\overline{X}_{2N_2}} + \overline{X}_{2N_2} - (\frac{\sigma^2}{\tau} + \tau)\right) + c\right]^2 I_{A^c}\right\}$$

where $c = (\frac{\sigma^2}{\tau} + \tau) - \frac{2(1-\rho_0)}{\rho_0} W_q$

$$\leq 2\Big(\tfrac{\rho_0}{2(1-\rho_0)}\Big)^2\left\{E\Big(\tfrac{s^2}{\overline{X}_{2N_2}}+\overline{X}_{2N_2}-(\tfrac{\sigma^2}{\tau}+\tau)\Big)^2 I_{A^c}+c^2 E I_{A^c}\right\}$$
$$= O(\tfrac{1}{b\alpha\omega_1}),\ \text{for any } \omega_1>1, \text{by (6.53).}$$

Let $B = \big[\overline{X}_{1N_1} \in D_1(\epsilon_0), \overline{X}_{2N_2} \in D_2(\epsilon_0)\big]$. For the first term, by Lemma 3, using arguments similar to (6.37) we have

$$E\big[\widehat{W}_q - W_q\big]^2 I_A = E\big[\widehat{W}_q - W_q\big]^2 I_B + O(\tfrac{1}{m_0^{\omega/2}}).$$

Using a Taylor expansion for the function $\widehat{W}_q$ on B, and noting that ω is arbitrary, it follows from (4.8), (4.11), and Lemma 6.2.2 that

$$\begin{aligned}
E\left[\frac{S^2+\overline{X}^2_{2N_2}}{2\big(\overline{X}_{1N_1}-\overline{X}_{2N_2}\big)}-\frac{\sigma^2+\tau^2}{2(\beta-\tau)}\right]^2 I_A &= \tfrac{(\sigma^2+\tau)^2}{4(\beta-\tau)^4}E(\overline{X}_{1N_1}-\beta)^2 \\
&\quad+\tfrac{(2\beta\tau-\tau^2+\sigma^2)^2}{4(\beta-\tau)^4}E(\overline{X}_{2N_2}-\tau)^2 \\
&\quad+\tfrac{1}{4(\beta-\tau)^2}E(s^2(N_2)-\sigma^2)^2 \\
&\quad+\tfrac{2\beta\tau-\tau^2+\sigma^2}{2(\beta-\tau)^3}E(\overline{X}_{2N_2}-\tau)(s^2(N_2)-\sigma^2) \\
&\quad+O(\tfrac{1}{b^{h_0}}) \\
&= V^0(b)+O(\tfrac{1}{b^{h_0}}).\ \square
\end{aligned}$$

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APPENDIX

TURBO PASCAL CODE FOR SIMULATIONS

```
program mu123(indata,outdata,seeddata);
{$N+}
(*****************************************************
 ** Written by Kevin Burns 30 October 1997          **
 ** This program implements the two-stage sampling  **
 ** plan for the function mu1(mu2 + mu3) and Bernoulli **
 ** populations.  The goal is to show that the MSE of  **
 ** sampling plan converges to the minimum MSE as the  **
 ** sampling budget tends to infinity.               **
 *****************************************************)

uses rvgen,crt;

var  X,Y,Z,i,j,n0,reps,mmore,nmore,lmore: integer;
     xbar,ybar,zbar,sigma1hat,sigma2hat,sigma3hat,fhat: real;
     sigma1,sigma2,sigma3,mstar,nstar,lstar,c1,c2,c3,Vest: real;
     bigmstar,bignstar,biglstar,mopt,nopt,lopt,V,ratio: real;
     mu1,mu2,mu3,f,fsqrsum,Vstdev,upper,lower,fdiff,bvhat: real;
     alpha,fsum: real;
     indata,outdata,seeddata: text;
     outfile: string[20];
     b,sumx,sumy,sumz,M,N,L: longint;

procedure startup;
(*** This procedure reads in the parameter values, gets the ***)
(*** initial random seeds, calculates the optimal          ***)
(*** allocation-and the minimum variance, and calculates the***)
(*** initial sample size.                                  ***)
begin  (* startup *)

  assign (indata,'musetup.dat');
  assign (seeddata,'seed.dat');
  reset (indata);
(* read in parameter values *)
  readln (indata,mu1,mu2,mu3,b,reps,c1,c2,c3);
  read (indata,outfile);
  assign (outdata,outfile);
  rewrite (outdata);
(* calculate initial sample size *)
  alpha:=(0.5+1-ln(3)/ln(b))/2;
  n0:= trunc(exp(alpha*ln(b)));
  writeln(outdata,'mu1 is ',mu1:6:4,' mu2 is ',mu2:6:4,' mu3 is
  ',mu3:6:4);
  writeln(outdata,'n0 is ',n0:4,' b is ',b:4,' alpha is
```

```
  ',alpha:6:4,' reps is ',reps:4);
  writeln(outdata,'c1 is ',c1:6:4,' c2 is ',c2:6:4,' c3 is
  ',c3:6:4);
  fsum:= 0;
  fsqrsum:= 0;
  f:= mu1*(mu2 + mu3);
  sigma1 := sqrt(mu1*(1-mu1));
  sigma2 := sqrt(mu2*(1-mu2));
  sigma3 := sqrt(mu3*(1-mu3));
(* calculate optimal allocation *)
  mopt:=(b*sigma1*(mu2+mu3))/(c1*sigma1*(mu2+mu3)+c2*mu1*sigma2+
        c3*mu1* sigma3);
  nopt:=(b*mu1*sigma2)/(c1*sigma1*(mu2+mu3)+c2*mu1*sigma2+c3*mu1
        *sigma3);
  lopt:=(b*mu1*sigma3)/(c1*sigma1*(mu2+mu3)+c2*mu1*sigma2+c3*mu1
        *sigma3);
  writeln(outdata,'mopt is ',mopt:8:6,' nopt is ',nopt:8:6,'
  lopt is ',lopt:8:6);
(* calculate minimum variance *)
  V:= (sqr(mu2+mu3)*sqr(sigma1))/mopt +
      (sqr(mu1)*sqr(sigma2))/nopt +(sqr(mu1)*sqr(sigma3))/lopt;
  reset(seeddata);
  getseeds(seeddata,3);
  close (indata);

end;  (* startup *)

procedure initsamp;
(*** This procedure takes the initial samples from the      ***)
(*** populations. Then, it estimates the means and standard***)
(*** deviations.                                             ***)
begin  (* initsamp *)

   sumx:=0;
   sumy:=0;
   sumz:=0;
   for i:= 1 to n0 do
     begin (* for loop to generate random variables  *)
       X := rvbernoulli(mu1,1);
         sumx:= sumx + X;
       Y := rvbernoulli(mu2,2);
         sumy:= sumy + Y;
       Z := rvbernoulli(mu3,3);
         sumz:= sumz + Z;
     end; (* for loop to generate random variables *)
(* estimate mus *)

if sumx = 0 then
  xbar:= 1/n0
  else
    if sumx = n0 then
      xbar:= (n0 - 1)/n0
        else
          xbar:= sumx/n0;
```

```
if sumy = 0 then
  ybar:= 1/n0
  else
    if sumy = n0 then
      ybar:= (n0 - 1)/n0
        else
          ybar:= sumy/n0;

if sumz = 0 then
  zbar:= 1/n0
  else
    if sumz = n0 then
      zbar:= (n0 - 1)/n0
        else
          zbar:= sumz/n0;

(* estimate sigmas *)
     sigma1hat := sqrt(xbar*(1-xbar));
     sigma2hat := sqrt(ybar*(1-ybar));
     sigma3hat := sqrt(zbar*(1-zbar));

end;  (* initsamp *)

procedure size;
(*** This procedure calculates the final sample sizes ***)
begin  (* size *)
if (sigma1hat+sigma2hat+sigma3hat) < 0.00001 then
  begin
    mstar :=b/2;
    nstar :=b/4;
    lstar :=b/4;
  end
else
  begin
  mstar:=(b*sigma1hat*(ybar+zbar))/(c1*sigma1hat*(ybar+zbar)+c2
         *xbar*sigma2hat+c3*xbar*sigma3hat);
  nstar:=(b*xbar*sigma2hat)/(c1*sigma1hat*(ybar+zbar)+c2*xbar
         *sigma2hat+c3*xbar* sigma3hat);
  lstar:=(b*xbar*sigma3hat)/(c1*sigma1hat*(ybar+zbar)+c2*xbar
         *sigma2hat+c3*xbar* sigma3hat);
  end;

   if mstar <= n0 then mmore:=0
     else mmore:=1;
   if nstar <= n0 then nmore:=0
     else nmore:=1;
   if lstar <= n0 then lmore:=0
     else lmore:=1;

   case mmore + 2*nmore + 4*lmore of
     1: begin   (* case 1: mstar gets the rest *)
          bigmstar:= (b-(c2+c3)*n0)/c1;
          bignstar:= n0;
```

```
        end;
     2: begin    (* case 2: nstar gets the rest *)
          bignstar:= (b-(c1+c3)*n0)/c2;
          bigmstar:= n0;
        end;
     3: begin    (* case 3: mstar and nstar split the rest *)
          bigmstar:= n0 + (mstar*(b-(c1+c2+c3)*n0))/(c1*mstar +
                     c2*nstar);
          bignstar:= n0 + (nstar*(b-(c1+c2+c3)*n0))/(c1*mstar +
                     c2*nstar);
        end;
     4: begin    (* case 4: lstar gets the rest *)
          bigmstar:= n0;
          bignstar:= n0;
        end;
     5: begin    (* case 5: mstar and lstar split the rest *)
          bigmstar:= n0 + (mstar*(b-(c1+c2+c3)*n0))/(c1*mstar +
                     c3*lstar);
          bignstar:= n0;
        end;
     6: begin    (* case 6: nstar and lstar split the rest *)
          bignstar:= n0 + (nstar*(b-(c1+c2+c3)*n0))/(c2*nstar +
                     c3*lstar);
          bigmstar:= n0;
        end;
     7: begin    (* case 7: they all get more *)
          bigmstar:= mstar;
          bignstar:= nstar;
        end;

   end;      (* end case statement *)
   biglstar:= (b - c1*bigmstar - c2*bignstar)/c3;

   M:= trunc(bigmstar);
   N:= trunc(bignstar);
   L:= trunc((b - c1*M - c2*N)/c3);

end;  (* size *)

procedure finalsamp;
(*** This procedure samples from the populations again  ***)
   var restx,resty,restz:integer;

   begin  (* finalsamp *)
     restx:= M - n0;
     resty:= N - n0;
     restz:= L - n0;

     if restx > 0 then
       begin   (* if restx > 0 *)
         for i:= 1 to restx do
           begin  (* for loop to generate X's *)
             X := rvbernoulli(mu1,1);
              if X = 1 then
```

```
              sumx:= sumx + 1;
          end;   (* for loop to generate X's *)
        xbar:= sumx/M;
      end;  (* if restx > 0 *)

      if resty > 0 then
      begin   (* if resty > 0 *)
        for i:= 1 to resty do
          begin  (* for loop to generate Y's *)
            Y := rvbernoulli(mu2,2);
             if Y = 1 then
              sumy:= sumy + 1;
          end;   (* for loop to generate Y's *)
        ybar:= sumy/N;
      end;  (* if resty > 0 *)

      if restz > 0 then
      begin   (* if restz > 0 *)
        for i:= 1 to restz do
          begin  (* for loop to generate Z's *)
            Z := rvbernoulli(mu3,3);
             if Z = 1 then
              sumz:= sumz + 1;
          end;   (* for loop to generate Z's *)
        zbar:= sumz/L;
      end;  (* if restz > 0 *)

    end;  (* finalsamp *)

procedure calculate;
(*** This procedure estimates the mean square error with a ***)
(*** confidence interval and then compares the estimated   ***)
(*** mean sqare error to the minimum variance.             ***)
  begin    (* calculate *)

    Vest:= fsum/reps;
    bvhat:= b*Vest;
    writeln(outdata,'The actual minimum MSE, V = ',V:10:8);
    writeln(outdata,'The estimated mean squared error, Vest =
    ',Vest:10:8);
    writeln(outdata,'b times the estimated mean squared error
    is ',bvhat:10:8);
    ratio:= Vest/V;
    writeln(outdata,'ratio is ',ratio:8:6);
    Vstdev:= sqrt((fsqrsum - (sqr(fsum)/reps))/(reps - 1));
    lower:= (Vest - (1.96*Vstdev/(sqrt(reps))))/V;
    upper:= (Vest + (1.96*Vstdev/(sqrt(reps))))/V;
    writeln(outdata,'95% lower and upper confidence limits
    for ratio are [',lower:10:8,          '  ',upper:10:8,']');
    rewrite (seeddata);
    for i:= 1 to 3 do
      writeln(seeddata,seed(i):12);
   end;  (* calculate *)
```

```
begin   (**** main program ****)
  startup;
     for j:= 1 to reps do
       begin   (* main for loop *)
         initsamp;
         size;
         finalsamp;
         fhat:= xbar*(ybar + zbar);
         fdiff:= sqr(fhat - f);
         fsum:= fsum + fdiff;
         fsqrsum:= fsqrsum + sqr(fdiff);
       end;  (* main for loop *)
  calculate;
  close(outdata);
  close (seeddata);
end. (* main program *)
```

```
program mg1(indata,outdata,seeddata);
{$N+}
(**********************************************************
 **  Written by Kevin Burns 28 February 1998               **
 **  This program implements the two-stage sampling plan for **
 **  the mean waiting time in the M/G/1 queue when the      **
 **  service times are exponential.  The goal is to show     **
 **  that the MSE of the sampling plan converges to the     **
 **  minimum MSE as the sampling budget tends to infinity.   **
 **********************************************************)
uses rvgen,crt;

var  i,j,n0,reps: integer;
     xbar,ybar,ysqrsum,ycubsum,y4sum,sigsqrhat,sub1,sub2,sub3,
     mstar,nstar,c1,c2,Vest,alpha,delta,phi,kappa3,kappa4,rho,
     bigmstar,bignstar,mopt,nopt,V,ratio,fsum,deltahat,phihat,
     beta,tau,sigmasqr,f,fsqrsum,Vstdev,upper,lower,fdiff,fhat,
     X,Y,sumx,sumy,sub1hat,sub2hat,kappa3hat,kappa4hat: real;
     indata,outdata,seeddata:text;
     outfile:string[20];
     b,M,N: longint;

procedure startup;
(*** This procedure reads in the parameter values, gets the ***)
(*** initial random seeds, calculates the optimal allocation***)
(*** and the minimum variance, and calculates initial sample***)
(*** size.                                                  ***)
begin  (* startup *)

  assign (indata,'setup.dat');
  assign (seeddata,'seed2.dat');
  reset (indata);
(* read in parameter values *)
  readln(indata,beta,tau,sigmasqr,kappa3,kappa4,rho,b,reps,
  c1,c2);
  read (indata,outfile);
  assign (outdata,outfile);
  rewrite (outdata);
(* calculate initial sample size *)
  alpha:=(0.5+1-ln(2)/ln(b))/2;
  n0:= trunc(exp(alpha*ln(b)));
  writeln(outdata,'beta is ',beta:6:4,' tau is ',tau:6:4,'
  sigma squared is ',sigmasqr:6:4);
  writeln(outdata,'kappa3 is ',kappa3:6:4,' kappa4 is
  ',kappa4:6:4,' rho is ',rho:6:4);
  writeln(outdata,'alpha is ',alpha:6:4,' n0 is ',n0:4,' b is
  ',b:4,' reps is ',reps:4);
  writeln(outdata,'c1 is ',c1:6:4,' c2 is ',c2:6:4);
  fsum:=0;
  fsqrsum:=0;
  f:= (sigmasqr+sqr(tau))/(2*(beta-tau));
  sub1:=2*beta*(kappa3*(3*sqr(tau)
        -sigmasqr)+tau*(kappa4+sigmasqr*(2*sqr(tau)-
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          3*sigmasqr)));
    sub2:=2*kappa3*tau*(sqr(tau)-sigmasqr)+kappa4*sqr(tau);
    sub3:=2*beta*tau-sqr(tau)+sigmasqr;
    delta:=sqrt(sqr(beta)*(4*kappa3*tau+kappa4+4*sqr(tau)*sigmasqr
           -sqr(sigmasqr))-sub1+sub2+sqr(sqr(tau))*sigmasqr
           -3*sqr(tau)*sqr(sigmasqr)+sigmasqr*sqr(sigmasqr));
    phi:=sqr(beta)*(4*kappa3*tau+kappa4
         -sqr(sqr(tau))+2*sigmasqr*(sqr(tau)-sigmasqr))
         -sub1+sub2+sigmasqr*(sqr(sqr(tau))
         -3*sqr(tau)*sigmasqr+sqr(sigmasqr));
  (* calculate optimal allocation *)
    mopt :=(b*beta*(sqr(tau)+sigmasqr)*(delta-
           beta*(sqr(tau)+sigmasqr)))/phi;
    nopt := b - mopt;
    writeln(outdata,'mopt is ',mopt:8:6,' nopt is ',nopt:8:6);
  (* calculate minimum variance *)
   V:= (sqr(sigmasqr+sqr(tau))*sqr(beta))/(4*sqr(sqr(beta
       -tau))*mopt)+(sqr(sub3)*sigmasqr)/(4*sqr(sqr(beta
       -tau))*nopt)+(kappa4-sqr(sigmasqr))/(4*sqr(beta
       -tau)*nopt)+(sub3*kappa3)/(nopt*2*(beta-tau)*sqr(beta
       -tau));
    reset(seeddata);
    getseeds(seeddata,2);
    close (indata);

  end;  (* startup *)

  procedure initsamp;
  (*** This procedure takes the initial samples from the     ***)
  (*** populations.  Then, it estimates the means and higher ***)
  (*** moments.                                               ***)
  begin  (* initsamp *)

  sumx:=0;
  sumy:=0;
  ysqrsum:=0;
  ycubsum:=0;
  y4sum:=0;

     for i:= 1 to n0 do
       begin (* for loop to generate random variables  *)
         X := rvexpon(beta,1);
           sumx:= sumx + X;
         Y := rvexpon(tau,2);
           sumy:= sumy + Y;
           ysqrsum:=ysqrsum + sqr(Y);
           ycubsum:=ycubsum + Y*sqr(Y);
           y4sum:=y4sum + sqr(sqr(Y));
       end; (* for loop to generate random variables *)
  (* estimate means *)
  xbar:= sumx/n0;
  ybar:= sumy/n0;
  (* estimate higher moments *)
  sigsqrhat := (ysqrsum-(sqr(sumy)/n0))/(n0-1);
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kappa3hat := ycubsum/n0 - 3*ysqrsum*ybar/n0 + 2*ybar*sqr(ybar);
kappa4hat := y4sum/n0 - 4*ycubsum*ybar/n0 +
             6*ysqrsum*sqr(ybar)/n0 - 3*sqr(sqr(ybar));

end;  (* initsamp *)

procedure size;
(*** This procedure calculates the final sample sizes ***)
begin  (* size *)

  sub1hat:= 2*xbar*(kappa3hat*(3*sqr(ybar)
            -sigsqrhat)+ybar*(kappa4hat+sigsqrhat*(2*sqr(ybar)
            -3*sigsqrhat)));
  sub2hat:= 2*kappa3hat*ybar*(sqr(ybar)
            -sigsqrhat)+kappa4hat*sqr(ybar);
  deltahat:= sqrt(sqr(xbar)*(4*kappa3hat*ybar+kappa4hat+4*
             sqr(ybar)*sigsqrhat-sqr(sigsqrhat))
             -sub1hat+sub2hat+sqr(sqr(ybar))*sigsqrhat
             -3*sqr(ybar)*sqr(sigsqrhat)+
             sigsqrhat*sqr(sigsqrhat));
  phihat:= sqr(xbar)*(4*kappa3hat*ybar+kappa4hat
           -sqr(sqr(ybar))+2*sigsqrhat*(sqr(ybar)-sigsqrhat))
           -sub1hat+sub2hat+sigsqrhat*(sqr(sqr(ybar))
           -3*sqr(ybar)*sigsqrhat+sqr(sigsqrhat));

  mstar:= (b*xbar*(sqr(ybar)+sigsqrhat)*(deltahat
          -xbar*(sqr(ybar)+sigsqrhat)))/phihat;

   if mstar <= n0
     then bigmstar:=n0
   else
     begin (* else *)
       if mstar >= ((b-c2*n0)/c1)
         then bigmstar:=((b-c2*n0)/c1)
       else  bigmstar:=mstar;
     end; (* else *)

   bignstar:= (b - c1*bigmstar)/c2;

   M:= trunc(bigmstar);
   N:= trunc((b - (c1*M))/c2);

end;  (* size *)

procedure finalsamp;
(*** This procedure samples from the populations again  ***)
   var restx,resty:integer;

   begin  (* finalsamp *)
     restx:= M - n0;
     resty:= N - n0;

     if restx > 0 then
       begin   (* if restx > 0 *)
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          for i:= 1 to restx do
            begin  (* for loop to generate X's *)
              X := rvexpon(beta,1);
                sumx:= sumx + X;
            end;  (* for loop to generate X's *)
          xbar:= sumx/M;
        end;  (* if restx > 0 *)

        if resty > 0 then
        begin   (* if resty > 0 *)

          for i:= 1 to resty do
            begin  (* for loop to generate Y's *)
              Y := rvexpon(tau,2);
                sumy:= sumy + Y;
                ysqrsum:=ysqrsum + sqr(Y);
                ycubsum:=ycubsum + Y*sqr(Y);
                y4sum:=y4sum + sqr(sqr(Y));
            end;  (* for loop to generate Y's *)
          ybar:= sumy/N;
          sigsqrhat := (ysqrsum-(sqr(sumy)/N))/(N-1);

        end;  (* if resty > 0 *)

      end;  (* finalsamp *)

procedure calculate;
(*** This procedure estimates the mean square error with a ***)
(*** confidence interval and then compares the estimated   ***)
(*** mean sqare error to the minimum variance.             ***)
  begin    (* calculate *)

      Vest:= fsum/reps;
      writeln(outdata,'The actual minimum MSE, V = ',V:10:8);
      writeln(outdata,'The estimated mean squared error, Vest =
      ',Vest:10:8);
      ratio:= (Vest-V)/V;
      writeln(outdata,'ratio is ',ratio:8:6);
      Vstdev:= sqrt((fsqrsum - (sqr(fsum)/reps))/(reps - 1));
      lower:= ((Vest-V) - (1.96*Vstdev/(sqrt(reps))))/V;
      upper:= ((Vest-V) + (1.96*Vstdev/(sqrt(reps))))/V;
      writeln(outdata,'95% lower and upper confidence limits for
      ratio are [',lower:10:8,         '  ',upper:10:8,']');
      rewrite (seeddata);
      for i:= 1 to 2 do
        writeln(seeddata,seed(i):12);
   end;  (* calculate *)

begin  (**** main program ****)
  startup;
      for j:= 1 to reps do
        begin   (* main for loop *)
          initsamp;
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          size;
          finalsamp;
          if ybar <= rho*xbar then
            fhat:= (sigsqrhat+sqr(ybar))/(2*(xbar-ybar))
          else
            begin
              fhat:= (rho*ybar*(sigsqrhat/sqr(ybar)+1))/(2*(1
                     -rho));
            end;
          fdiff:= sqr(fhat - f);
          fsum:= fsum + fdiff;
          fsqrsum:= fsqrsum + sqr(fdiff);
        end;  (* main for loop *)
  calculate;
  close(outdata);
 close (seeddata);
end. (* main program *)
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KEVIN EDWARD BURNS
Optimal Two Stage Procedures for Estimating Functions of Parameters in Reliability and Queueing Models
(Under the direction of ANDREW F. SEILA)

In this dissertation, we consider the problem of estimating functions of parameters found in reliability and queueing models. The problem is to allocate a fixed sampling budget among the populations with the goal of minimizing the mean squared error (MSE) of the estimator. We consider the reliability model with three components such that the probability the system works is $f(\mu_1, \mu_2, \mu_3) = \mu_1(\mu_2 + \mu_3)$, and the mean waiting time of the M/G/1 queue. For each of these models, we consider a set of sample sizes referred to as a first-allocation procedure which minimizes the first-order approximation to the MSE. Since the first-order allocation procedure depends on the unknown parameters in the model, we propose a two-stage procedure in which we first use a fraction of the sampling budget to estimate the unknown parameters and then allocate the remaining budget based on the initial sample. We show that the difference between the MSE for the two-stage procedure and the minimum MSE obtained using the optimal set of sample sizes from the first-allocation procedure goes to zero as the budget goes to infinity. Simulations are used to demonstrate the asymptotic optimality results for the two stage procedures. The empirical studies show that the two stage estimation procedures work well for reasonable sample sizes.

INDEX WORDS: Allocation, Asymptotic, Distribution, Estimator, Exponential, Infinite, Mean squared error, Model, Optimal, Parameters, Queue, Sample Size, Simulation